Designing and developing image editing tools in MATLAB using intuitionistic fuzzy sets

FINAL REPORT
OF THE WORK DONE ON THE MAJOR RESEARCH PROJECT

Approval No. and date

F. No: 43 - 418/2014 (SR), dated September 2015

Submitted to

ज्ञान-विज्ञान विमुक्तये

UNIVERSITY GRANTS COMMISSION
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November 2018

ACKNOWLEDGEMENT

First of all, I thank University Grants Commission for its financial support to carry out this major research project entitled "Designing and developing image editing tools in MATLAB using intuitionistic fuzzy sets".

I express my sincere thanks to Thiru.S.D.Chandrasekar, Secretary and Correspondent, Vellalar College for Women for his support to carry out this research work.

I am very much grateful to Dr.(Mrs).D.Kamalaveni, Former Principal, Vellalar College for Women, Erode for her patronage and co-operation extended to me for doing my research work in the institution.

I express my deep sense of gratitude to Dr.(Mrs).N.Maragatham, Principal, Vellalar College for Women, Erode for her able guidance and valuable advice throughout this research.

I express my sincere thanks to all the faculty members of the Department of Mathematics, Vellalar College for Women for their support and timely help during the period of research.

I thank Ms.C.Yuvapriya, Project Fellow of this project for her constant support during the research period.

My wholehearted thanks to my Ph.D Scholars Dr.K.K.Myithili, Dr.G.Thamizhendhi, Mrs.C.Radhika, Mrs.C.Radhamani and M.Phil Scholar Mrs.K.Indhurekha for their support and contributions to this research project.

Words are inadequate to express my profound feeling of gratitude to my family members and friends, for their moral support and affection at all stages of my life.

Above all, I thank and praise the God almighty for giving me the spirit, moral strength and courage for the successful completion of this project.

Contents

Abstract

The main emphasis reported in this report is to provide intuitionistic fuzzy logic tools for handling the data with uncertainty. On the foundation of the theory of intuitionistic fuzzy sets, this work extends the traditional research by presenting new definitions and desirable properties of intuitionistic fuzzy statistical tools also.

The work starts with defining intuitionistic fuzzification and intuitionistic defuzzification functions which are the important components of intuitionistic fuzzy logic controller (IFLC). An architecture of IFLC is proposed and its capability is clearly elucidated through suitable illustrations.

Further, new intuitionistic fuzzy operators are defined and applied in intuitionistic fuzzy inference engine to establish a flexible mathematical framework to model the vagueness, which will find applications in noise removal in image processing. Comparitive analysis of intuitionistic fuzzy filters with traditional and fuzzy filters is done and the experimental results illustrate the validity of the proposed technique.

Intuitionistic fuzzy random variable (IFRV) is defined and some of its interesting properties are studied. The expectation, variance and moments of IFRV are discussed. Intuitionistic fuzzy number is defined as a generalization of Wu's fuzzy

number. In addition, intuitionistic fuzzy statistical tools like mean, median, mode for intuitionistic fuzzy data are explained through suitable illustrations which are helpful in designing intuitionistic fuzzy filters.

Multi-period decision making model is proposed using intuitionistic fuzzy moving aggregation operator under uncertain environment and compared with existing crisp and fuzzy moving average operators. The validity of the proposed technique is verified with the economical time series data, to forecast the gross domestic product (GDP) growth of Indian economy.

The concepts of distance, eccentricity, radius, diameter and center of an intuitionistic fuzzy tree are defined. Some of the domination parameters like independent domination, connected domination and total domination on intuitionistic fuzzy trees are investigated. The procedure for intuitionistic fuzzification for numerical data set is proposed. Further, intuitionistic fuzzy tree center-based clustering algorithm is designed. The effectiveness of the algorithm is checked with a numerical dataset and compared with two existing clustering methods.

A hypergraph is a set V of vertices and a set E of non-empty subsets of V, called hyperedges. Unlike graphs, hypergraphs can perform higher-order interactions in social and communication networks. Directed hypergraphs are much like directed graphs. Colors are used to distinguish the classes. Coloring a hypergraph H must assign atleast two different colors to the vertices of every hyperedge. That is, no edge is monochromatic. Here, p-coloring, $\mathcal{K}$-coloring, p-chromatic number, spike and spike reduction of intuitionistic fuzzy directed hypergraph (IFDHG), skeleton of spike reduction are studied. Further, a few properties of coloring of IFDHG are discussed. Also, it has been proved that in an ordered IFDHG, a primitive coloring is a $\mathcal{K}$-coloring of the IFDHG.

Upper and lower truncation, core aggregate of IFDHG, conservative $\mathcal{K}$-coloring of IFDHG, chromatic values of intuitionistic fuzzy colorings, elementary center of intuitionistic fuzzy coloring, f-chromatic value of intuitionistic fuzzy coloring, intersecting IFDHG, $\mathcal{K}$-intersecting IFDHG, strongly intersecting IFDHG were studied. Also it has been proved that IFDHG H is strongly intersecting if and only if it is $\mathcal{K}$-intersecting.

Essentially intersecting, essentially strongly intersecting, skeleton intersecting, non-trivial, sequentially simple and essentially sequentially simple IFDHGs are defined. Also, it has been proved that if IFDHG H is ordered and essentially intersecting, then $\chi(H) \leq 3$. An IFDHG H is strongly intersecting if and only if $H^{\langle r_i, s_i \rangle}$ is intersecting for every $\langle r_i, s_i \rangle \in F(H)$ is proved and an application of IFDHG in molecular structure representation is also given.

Fuzzy sets and intuitionistic fuzzy sets handle uncertainty and vagueness which Cantorian sets could not handle. Temporal intuitionistic fuzzy set with a time domain is an extension of intuitionistic fuzzy set and is useful in dealing with uncertainty and vagueness present in the time dependent real environment. A new type of intuitionistic fuzzy set called multi-parameter temporal intuitionistic fuzzy set is proposed and it's operations are defined. Further, extended triangular membership and non-membership functions for temporal intuitionistic fuzzy sets and multi-parameter temporal intuitionistic fuzzy sets are defined. Geometric interpretation of a temporal intuitionistic fuzzy set is also dealt with a suitable example.

Finally, a study on Indian Universities Ranking using InterCriteria Decision

Making (ICDM) Method is given. This approach is used to real data extracted from Indian University Ranking System for the year 2017 by National Institutional Ranking Framework (NIRF). The NIRF provides for ranking of institutes in five broad generic parameters, namely: i) Teaching, Learning and Resources; ii) Research and Professional Practice; iii) Graduation Outcome; iv) Outreach and Inclusivity; and v) Perception. The aim is to analyze the correlation between the above-said parameters in the Ranking System.

Chapter 1

Introduction

This chapter is an exhaustive review of the literature on sets, fuzzy sets, intuitionistic fuzzy sets, fuzzy logic controller and intuitionistic fuzzy logic controller, intuitionistic fuzzy logic tools. Some basic definitions and terminology which are used in constructing the properties relating to the report are given.

1.1 Fuzzy sets

A *crisp set* is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those which certainly belong to the set) and non-members (those which certainly do not). A sharp unambiguous distinction exists between the members and non-members of the set [45].

A set is described by a function, usually called a *characteristic function*, that declares which elements of X are members of the set and which are not. That is, the characteristic function of a set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and non-members of the set under consideration.

Mathematically, the characteristic function of a set A maps elements of the universal set X to the two-valued set $\{0,1\}$, which is formally expressed by $\chi_A : X \to \{0,1\}$. For each $x \in X$, when $\chi_A(x) = 1$, x is declared to be a member of A; when $\chi_A(x) = 0$, x is declared to be a non-member of A.

Gone are the days of two-valued logic and now this is an era of multi-valued logic, where characteristic function may fail to deal situations with ambiguity. For example, black or white can have many meanings. Grey is neither black nor white but in between both. Hence, imprecision plays an important role in information representation in real process where the increase in precision would otherwise become unmanageable.

Lotfi A. Zadeh founded fuzzy sets in 1965, to make set theory more intuitive and applicable to the real world problems [110]. A *fuzzy set* is any set that allows its elements to have different degree of membership in the interval [0, 1]. As fuzzy set is an extraordinary tool for representing human knowledge and perception, it has achieved successful applications in various fields of fuzzy decision-making, approximate reasoning, statistics with imprecise probabilities and fuzzy control [38, 85].

Two years after the concept of fuzzy set was proposed, L-fuzzy set was developed by Gogeun as a generalization of fuzzy sets. Nevertheless, in 1973, Zadeh himself established that knowledge can be better represented by means of some generalizations of fuzzy sets. Thus, the extensions of fuzzy set theory arise in this way. One such extension is on statistical methods.

Currently, there are more researches focusing on the fuzzy statistical analysis and applications [30, 35, 40, 91]. New statistical approach for fuzzy data was given by Ching-Min Sun and Berlin Wu [36]. Wu and Sun illustrated about interval-valued statistics, fuzzy logic and its applications. Liu illustrated uncertainty theory based on fuzzy set. Fuzzy mean, fuzzy median, fuzzy mode were defined by Hung T. Nguyen and Berlin Wu [71].

Cornelis et al. [38] highlighted the applications of fuzzy techniques in image processing. Mike Nachtegael, Dimitri Van de Ville, Etienne E. Kerre , Russo [70, 90] introduced fuzzy filters and its extension.

Timothy J. Ross [100] developed fuzzy decision making by describing basic concepts in multi objective decisions. Imprecise information are aggregated by fuzzy OWA operators, several extensions and generalizations were studied [105–108]. Xu defined fuzzy harmonic mean operators and presented an approach to multiple attribute group decision making with a practical example [103].

Merigo proposed decision making with fuzzy moving averages and OWA operators [64]. Merigo further extended and generalized moving average using distance measure [63]. Fuzzy ordered weighted moving average is introduced and studied in [61, 62] and aggregation operators between the fuzzy minimum and the fuzzy maximum in a dynamic process is defined with the moving average. Properties are studied and a wide range of particular cases are presented including fuzzy moving average, fuzzy moving maximum and fuzzy moving minimum in [59, 60].

1.2 Fuzzy logic

Fuzzy logic is an approach to compute the values based on "degrees of truth" rather than the usual "true or false" boolean logic. In other words, an organized method for dealing imprecise data is called fuzzy logic, where the data are considered as fuzzy sets. Fuzzy logic is basically a multivalued logic that allows values between conventional evaluations like yes or no, true or false, black or white, etc. Zadeh introduced the term *fuzzy logic* in his seminal work "Fuzzy sets", which describes the mathematics of fuzzy set theory (1965) [45].

In other words, fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness. Fuzziness in the fuzzy set is characterized by its membership function, which represents the degree of truth in fuzzy logic.

Membership functions

A membership function is a curve that defines how each point in the input space is mapped to a membership value between 0 and 1 [94]. Membership functions have different shapes, such as the triangular, trapezoidal, gaussian, bell-shaped, sigmoidal, S-curve, Z-curve and Pi-curve. Defining appropriate membership function is an important task for any actual application using fuzzy logic. For the systems with variation in a short interval of time, a triangular or trapezoidal curve can be selected and for the system with very high control accuracy, gaussian or S-curve curve can be utilized. Most frequently used membership functions are *triangular*, *trapezoidal*, *Gaussian*, *bell-shaped*, *sigmoidal* and simplest among them are formed using straight lines.

Another important concept in fuzzy logic is IF-THEN rule. The role of fuzzy logic is to map an input space to an output space, and the primary mechanism for doing this is a list of IF-THEN statements called *rules*. In fuzzy logic, mapping rules can be specified in terms of words rather than numbers. *Computing with words* explores imprecision and tolerance.

IF-THEN Rules

Fuzzy sets and fuzzy operators are the subjects and verbs of fuzzy logic. These IF-THEN rule statements are used to formulate the conditional statements that comprise fuzzy logic [117]. A single fuzzy IF-THEN rule assumes the form

$$\textit{If } X \textit{ is } A \textit{ then } Y \textit{ is } B$$

where A, B are linguistic values defined by fuzzy sets on the ranges (universes of discourse) X and Y respectively. The IF-part of the rule X is A, called the *antecedent* or premise, while THEN-part of the rule Y is B, called the *consequent* or conclusion. An example of such a rule might be

If education is good then salary is high.

Note that *good* is represented as a number between 0 and 1, and so the antecedent is an interpretation that returns a single number between 0 and 1. On the other hand, *high* is represented as a fuzzy set, and so the consequent is an assignment that assigns the entire fuzzy set B to the output variable Y.

Fuzzy operators

In fuzzy logic, operators such as AND, OR and NOT are implemented by intersection (min), union (max) and complement operators and are applied in various control applications like air-conditioner, anti-braking system in vehicles, control on traffic lights and washing machines.

1.3 Fuzzy logic controller

The basic idea behind *fuzzy logic controller* (FLC) is to incorporate the "expert experience" of a human operator in the design of the controller in controlling a process whose input - output relationship is described by collection of fuzzy control rules (e.g. IF-THEN rules) involving linguistic variables, rather than a complicated dynamic model. FLC is the best utilized controller in complex ill-defined process that can be controlled by a skilled human operator without much knowledge of their underlying dynamics [58].

The FLC provides superior results to those obtained by conventional control systems and appears very useful for complex problems whose available information are interpreted qualitatively, inexactly, or uncertainly.

A typical architecture of FLC is shown in Figure 1.1, comprises of a fuzzifier, fuzzy rule base, inference engine and a defuzzifier.

Fuzzification

Fuzzification is the process of making a crisp quantity into fuzzy. In real life applications, due to the inevitable measurement and inaccuracy, the exact values of the measured quantities are not known and so the problem involved in that situation are usually defined in an uncertain way. The uncertainty happens to arise because of imprecision, ambiguity or vagueness and can be represented by a membership function.

Fuzzification determines the degree of membership. In order to express the system inputs in linguistic terms, fuzzification process is used, so that the rules

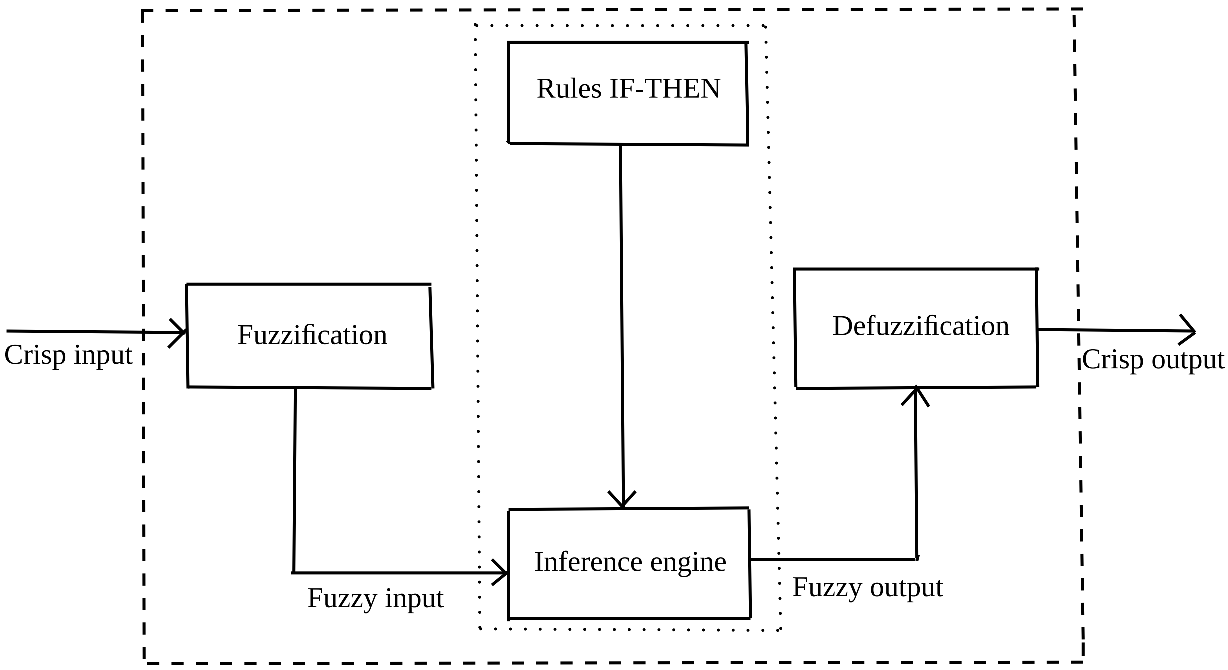

Figure 1.1: Fuzzy logic controller

can be applied in a simple manner to express a complex system. In fuzzy control applications, the observed data is crisp. Therefore, fuzzification is necessary during an earlier stage to manipulate the data in FLC.

Fuzzy Rules

Human beings make decisions on rules. Knowingly or unknowingly all the decisions are based on computer like IF-THEN statements.

Example

If the weather is bad, then decide not to go out.

If the forecast says, the weather will be bad today, but fine tomorrow, then decision would be not to go today and postpone it till tomorrow.

Rules associate ideas and rotate one event to another. Fuzzy machines mimic the human behaviour and work in the same manner, where the decision are replaced by fuzzy sets and the rules are replaced by fuzzy rules. In a FLC, a rule base is constructed to control the output variable. A *fuzzy rule* is a simple IF-THEN rule with a condition and a conclusion [52].

Fuzzy Inference Systems

Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic. The mapping provides a basis from which decisions can be made. The process of fuzzy inference involves : "membership functions, logical operations and IF-THEN rules".

Types of Fuzzy inference systems

There are two types of fuzzy inference systems: Mamdani-type and Sugeno-type.

- Mamdani fuzzy system is based on fuzzy MAX-MIN operator.
- Sugeno fuzzy system are used to develop a systematic approach in creating and analyzing the fuzzy system and for sample data-based fuzzy modeling.

Mamdani Fuzzy Systems

The Mamdani fuzzy systems are fuzzy systems with fuzzy IF-THEN rules in the form of

$$\text{If } X \text{ is } P \text{ and } Y \text{ is } Q \text{ and} \ldots \text{THEN } Y \text{ is } R$$

where P, Q, $\ldots$ and R are fuzzy sets.

Sugeno Fuzzy Systems or Takagi Sugeno Fuzzy System (TSK)

TSK fuzzy systems are fuzzy systems with IF-THEN rules in the form

$$\text{If } X \text{ is } P \text{ and } Y \text{ is } Q \text{ THEN } Z = f(X, Y)$$

where P and Q are fuzzy set in the antecedent, while $Z = f(X, Y)$ is a crisp function in the consequent.

Defuzzification

Defuzzification means the fuzzy-to-crisp conversions. In general, both the inputs and outputs of a fuzzy inference engine are fuzzy variables. The fuzzy results generated cannot be used to the real time applications and it is necessary to convert the fuzzy quantities into crisp quantities for further processing. So the fuzzifier and defuzzifier blocks are needed to accept crisp inputs and produce crisp outputs.

1.4 Intuitionistic fuzzy sets

In real life, due to the insufficiency in the availability of information, the evaluation of membership values is not possible upto our satisfaction. Therefore, a generalization of fuzzy sets was introduced by Krassimir T Atanassov (1983) as intuitionistic fuzzy set (IFS), which include both membership and non-membership of an element in the set. However, in reality, a part remains indeterministic on which hesitation survives which is called as intuitionistic fuzzy index. In such situations, IFSs seem to be applicable to address the issues of uncertainty. In case, when the degree of rejection is defined simultaneously with the degree of acceptance and when both these degrees are not complementary to each other, then IFS can be used as a more general tool for describing uncertainty.

IFSs have got their applications in different fields such as image processing [74, 98], statistics [76], decision-making problems [16, 56, 57], pattern recognition and so on. Among these, the works of Krassimir T Atanassov [10–12, 17, 18], Krassimir T Atanassov and Gargov [15], Szmidt and Kacprzyk [96, 97] are noteworthy. The notion of vague set is the same as that of IFS, is pointed out by Bustince and Burillo [32]. In 1993, Gau and Buehrer [46] introduced the concept of vague sets which is another generalization of fuzzy sets.

Ioannis et al. [48] initiated an attempt towards intuitionistic fuzzy image processing and presented an intuitive approach for intuitionistic fuzzification of images. Ioannis and George have worked on IF contrast enhancement. Tamalika Chaira [98] have proposed a new method for IF segmentation and edge detection of medical images. Parvathi et al. have developed an algorithm on intuitionistic fuzzy approach for image enhancement using contrast intensification operator [73]. Also some attempt was made to define theoretical concepts in IF statistical for filters by Parvathi et al. is an initiative to define theoretical concepts [75].

Buhaescu defined @ operator [27] and Atanassov introduced $G_{(\alpha,\beta)}$ operator and some of its properties. Ranjit Biswas and Roy gave some operations on IFSs [39]. Angelov have defined defuzzification over intuitionistic fuzzy sets in [6].

1.5 Motivation

The design of *traditional logic controller* usually requires a mathematical model of the process involved. The construction of such model is difficult for many real world applications due to partial information. The imprecise description of the problem can be handled as an alternative approach by expert human operators. This modeling leads to the usage of fuzzy concepts which is close to human perception than traditional logical system.

Data given as inputs to a fuzzy logic system and data used for tuning, are often noisy, thus bearing an amount of uncertainty. Designing of FLC involves definition of fuzzy sets. FLC for noisy images is designed using fuzzy filtering techniques applied in inference engine.

Many research works are going in filtering techniques from non-fuzzy to fuzzy. Traditional statistical filters are defined in digital image processing by Gonzalez [84]. In literature, several authors were working on fuzzy filters and their extensions. Pal et al. worked on image enhancement using fuzzy set and designed an algorithm using contrast intensification operator [72]. Sharmistha Bhattacharya et al. proposed a contrast removal fuzzy operator in image processing [92]. C.V. Jawahar developed fuzzy statistics for digital images which is an alternative representation to hard statistics [51]. Farizio Russo presented an overview of non-linear fuzzy filters on their similarities and differences [90].

Mike Natchtegael et al. presented an overview of existing classical and fuzzy filters for noise reduction and a comparison study has been reported in [70]. Ville. D. Nachtegael, D. Weken, E. Kerre, W. Philips and I. Lemahieu introduced a two staged noise reduction technique with additive noise by fuzzy image filtering [101].

J. Sorubal Marcel et al. proposed fuzzy approach for detecting and removing salt and pepper noise. Jagadish H. Pujar [50] described a robust fuzzy median filter for impulse noise reduction of gray scale images. Bhavana Deshpande designed a fuzzy based median filtering for noise removal along with the fuzzy rule based approach to improve the filter performance in salt-and-pepper noise detection and cancellation [26].

Among extensions of fuzzy sets, IFSs [9, 15] provide a intuitive framework to deal vagueness from imprecise information by considering non-membership values in addition to membership values. IFS plays an important role in the field of engineering, statistics, graph theory, signal processing, medical diagnosis [49], pattern recognition, decision making and expert system. Agarwal et al. presented design of a probabilistic intuitionistic fuzzy rule based controller. Akram et al. developed intuitionistic fuzzy logic (IFLC) for heater fans [4] and washing machines [5]. Thus, so far IFLCs are designed to specific applications.

Designing of a common IFLC is not found anywhere in the literature. Hence authors are motivated to design an IFLC using intuitionistic fuzzification and defuzzification functions. Moreover, intuitionistic fuzzy filtering techniques in images are considered to verify the effectiveness of the proposed IFLC.

IFS theory provides a tool to derive filters for image processing in intuitionistic fuzzy (IF) environment. IF statistical tools like mean, median, mode for IF data defined by Parvathi et al. [75, 76] is very helpful for developing IF filters in image processing. Further, the authors have applied IF operators in intuitionistic fuzzy inference engine to establish a flexible mathematical framework for image processing.

Also, application of IF statistics is more futuristic and are widely used in decision making problems [113]. Hence, IF statistical tools and IF averaging operators are also introduced and studied.

1.6 Organisation of the Report

The title of the report is "Designing and developing image editing tools in MATLAB using intuitionistic fuzzy sets" is divided into ten chapters. The summary of each chapter is as follows:-

Chapter I gives a brief introduction, review of literature, basic definitions and terminology relating to this study.

Chapter II deals with fuzzification and defuzzification of intuitionistic fuzzy sets. The term *intuitionistic fuzzification function* refers to formulating membership and non-membership functions of an IFS. An attempt has been made to introduce various types of intuitionistic fuzzification functions such as triangular, trapezoidal, Gaussian, bell-shaped, sigmoidal, S-shaped, Z-shaped functions which are more useful in modeling real world situations in IF environment.

IF-defuzzification function is a function used to convert membership and non-membership values into precise quantity. The term *IF-defuzzification function* (IFDF) refers to formulation of defuzzification function of an IFS. IF-defuzzification functions such as triangular, trapezoidal, L-trapezoidal, R-trapezoidal, gaussian, S-shaped, Z-shaped functions are defined. The proposed defuzzification techniques are useful to develop IFLC.

Chapter III is devoted to develop a common architecture of IFLC. The capability of the proposed architecture is clearly elucidated through the experimental results. Further, IF operators are applied in intuitionistic fuzzy inference engine to establish a flexible mathematical framework for image processing. Comparitive analysis of intuitionistic fuzzy filters with traditional and fuzzy filters is done, experimental results illustrate the validity of the proposed technique and provided that intuitionistic fuzzy filter gives better performance.

In Chapter IV, intuitionistic fuzzy random variable is defined and some of its properties are discussed. Also, characteristics of intuitionistic fuzzy random variable like expectation, variance and moments are described. Intuitionistic fuzzy number is defined as a generalization of Wu's fuzzy number. Statistical tools are defined for intuitionistic fuzzy data and explained through suitable illustrations. In addition, IF statistical tools are described which are helpful in designing intuitionistic fuzzy filtering algorithm.

In this chapter, an initiation is taken to model the vagueness associated with the image which will find applications in noise removal in image processing. The proposed algorithm removes the noise in the image and improves the image quality without any loss of edge information. The performance of the proposed method is tested in MATLAB simulations for an image that has been subjected to various noises. Performance analysis is done on the basis of statistical measures like correlation coefficient, peak signal to noise ratio and mean square error values.

In Chapter V, relation between intuitionistic fuzzy ordered weighted averaging operator and $G_{\alpha,\beta}$ operator are introduced. Moreover, based on the newly defined relations IF set theory paves way for the introduction of several IF moving averages like IF weighted moving average, IF ordered weighted moving average, IF ordered weighted averaging-weighted moving average, IF induced ordered weighted moving average, IF weighted geometric moving average and IF weighted harmonic moving average. Further, it is extended by using distance measures suggesting the concept of IF moving average distance, IF ordered weighted moving average distance and the IF induced ordered weighted moving average distance operator. Also, a model is proposed using intuitionistic fuzzy moving aggregation operator under multi-period decision making with uncertainty and compared with other existing crisp techniques. The validity of the proposed technique is verified with the economical time series data, to forecast the gross domestic product of Indian Economy. The study shows that intuitionistic fuzzy moving aggregation operator is a better tool to reflect the original forecast.

In Chapter VI, the concepts of distance, eccentricity, radius, diameter and center of an intuitionistic fuzzy tree are defined. Some of the domination parameters like independent domination, connected domination and total domination on intuitionistic fuzzy trees are investigated. The procedure for intuitionistic fuzzification for numerical data set is proposed. Further, intuitionistic fuzzy tree center-based clustering algorithm is designed. The effectiveness of the algorithm is checked with a numerical dataset and compared with two existing clustering methods.

In Chapter VII, p-coloring, $\mathcal{K}$-coloring, p-chromatic number, spike and spike reduction of intuitionistic fuzzy directed hypergraph (IFDHG), skeleton of spike reduction are studied. Further, a few properties of coloring of IFDHG are discussed. Also, it has been proved that in an ordered IFDHG, a primitive coloring is a $\mathcal{K}$-coloring of the IFDHG.

Upper and lower truncation, core aggregate of IFDHG, conservative $\mathcal{K}$-coloring of IFDHG, chromatic values of intuitionistic fuzzy colorings, elementary center of intuitionistic fuzzy coloring, f-chromatic value of intuitionistic fuzzy coloring, intersecting IFDHG, $\mathcal{K}$-intersecting IFDHG, strongly intersecting IFDHG were studied. Also it has been proved that IFDHG H is strongly intersecting if and only if it is $\mathcal{K}$-intersecting.

In Chapter VIII, essentially intersecting, essentially strongly intersecting, skeleton intersecting, non-trivial, sequentially simple and essentially sequentially simple IFDHGs are defined. Also, it has been proved that if IFDHG H is ordered and essentially intersecting, then $\chi(H) \leq 3$. An IFDHG H is strongly intersecting if and only if $H^{\langle r_i, s_i \rangle}$ is intersecting for every $\langle r_i, s_i \rangle \in F(H)$ is proved and an application of IFDHG in molecular structure representation is also given.

In Chapter IX, a new type of intuitionistic fuzzy set called multi-parameter temporal intuitionistic fuzzy set is proposed and it's operations are defined. Further, extended triangular membership and non-membership functions for temporal intuitionistic fuzzy sets and multi-parameter temporal intuitionistic fuzzy sets are defined. Geometric interpretation of a temporal intuitionistic fuzzy set is also dealt with a suitable example.

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Research and Professional Practice; iii) Graduation Outcome; iv) Outreach and Inclusivity; and v) Perception. The aim is to analyze the correlation between the above-said parameters in the Ranking System.

Chapter 2

Fuzzification and defuzzification of intuitionistic fuzzy sets

2.1 Introduction

Intuitionistic fuzzification functions provide a flexible model to eloborate uncertainty and vagueness involved in real world problems. In order to employ intuitionistic fuzzy logic, operations, rules in mathematical modeling, it is necessary to convert crisp values into intuitionistic fuzzy pairs (IFPs) [1]. This process is known as *intuitionistic fuzzification*. In this chapter, several types of membership and non-membership functions with hesitancy index as an arbitrary parameter for triangular, trapezoidal, Gaussian, bell-shaped, sigmoidal, S-shaped, Z-shaped functions characterizing IFSs are defined and studied. Hence, an attempt has been made to formulate fuzzification and defuzzification functions for IFSs.

Appropriate intuitionistic fuzzy membership function for a specific purpose can be selected and applied.

[1]Refer Page 91 for the mathematical definition of IFPs

2.2 Preliminaries

In this section, some basic definitions, which are pre-requisites for the study, are outlined.

Definition 2.2.1. [110]

Let X be a nonempty set. A *fuzzy set* $\tilde{A}$ drawn from X is an object of the form

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x) : x \in X\}$$

where $\mu_{\tilde{A}} : X \to [0, 1]$ is the membership function of the fuzzy set $\tilde{A}$.

Example 2.2.1. Let X=$\{20, 21, 25, 26, 28, 81\}$ be the universe of discourse representing people's age in a house. Let $\tilde{A}$ be the fuzzy set representing the *youth* people.

$$A = \{(20, 1), (21, 0.9), (25, 0.5), (26, 0.40), (28, 0.20), (81, 0)\}$$

Here $\tilde{A}$ indicates that the person with age 20 is an exact member of the set youth with maximum grade 1 and the person with age 21 is also a member of grade 0.9 and so on. Note that a person whose age is 81 is not at all a member of the *youth* set.

Definition 2.2.2. [9]

Let the universal set X be fixed. An *intuitionistic fuzzy set* A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degrees of membership and non-membership of the element $x \in X$ respectively, and for every $x \in X$ in A, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ holds.

Note

(i) *Membership function* for an intuitionistic fuzzy set A on the universe of discourse X is defined as $\mu_A : X \to [0,1]$, where each element x is mapped to a value between 0 and 1. The value $\mu_A(x)$ is called the membership value or degree of membership of the element x in the IFS A.

(ii) *Non-membership function* for an intuitionistic fuzzy set A on the universe of discourse X is defined as $\nu_A : X \to [0,1]$, where each element x is mapped to a value between 0 and 1. The value $\nu_A(x)$ is called the non-membership value or degree of non-membership of the element x in the IFS A.

Definition 2.2.3. [9]

For every common intuitionistic fuzzy subset A on X, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the *intuitionistic fuzzy index or hesitancy index* of x in A.

Note

$\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A. $\pi_A(x)$ expresses the degree of lack of knowledge of every $x \in X$ belongs to IFS or not. Obviously, for every $x \in X$, $0 \leq \pi_A(x) \leq 1$.

2.3 Intuitionistic fuzzification functions

The term *intuitionistic fuzzification functions* refers to formulation of membership and non-membership functions of an IFS. In IFS, there are many ways to characterize fuzziness to depict the membership and non-membership functions graphically. The choice of which of the methods to be used depends entirely on the problem under consideration. The graphical representations may include different shapes formed using straight lines and simple curves. The formulated membership and non-membership functions themselves can take any form like triangles, trapezoids, bell curves or any other shape as long as those shapes accurately represent the distribution of information within the system.

The simplest membership and non-membership functions are formed using straight lines. Among these, *intuitionistic fuzzy triangular functions* are formed by the collection of three points forming a triangle and *intuitionistic fuzzy trapezoidal functions* are just a truncated triangle curve with a flat top.

The *intuitionistic fuzzy Gaussian* and *bell-shaped functions* are formed by smooth curves and *intuitionistic fuzzy sigmoidal functions* are also simple curves which is either open left or right. *Intuitionistic fuzzy S-shaped* and *Z-shaped functions* are formed by polynomial based curves.

This section discusses the formulation and the features of the above-mentioned intuitionistic fuzzy functions. Suitable illustrations are also dealtwith. Throughout this chapter, A represents an *intuitionistic fuzzy set* and $x \in A$.

2.3.1 Intuitionistic fuzzy triangular function (*iftrif*)

The intuitionistic fuzzy triangular function *iftrif*, is specified by three parameters, a lower limit a, an upper limit c, and a value b, where $a \leq b \leq c$. The precise appearance of the function is determined by the choice of the parameters a, b, c which in turn forms a triangle. In this a and c locates the *feet* of the triangle and the parameter b locates the *peak*.

Intuitionistic fuzzy triangular membership function of A takes the form

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq a \\ (\frac{x-a}{b-a}) - \epsilon & ; \quad a < x \leq b \\ (\frac{c-x}{c-b}) - \epsilon & ; \quad b \leq x < c \\ 0 & ; \quad x \geq c \end{cases}$$

The corresponding intuitionistic fuzzy triangular non-membership function is of the form

$$\nu_A(x) = \begin{cases} 1 - \epsilon & ; \quad x \leq a \\ 1 - (\frac{x-a}{b-a}) & ; \quad a < x \leq b \\ 1 - (\frac{c-x}{c-b}) & ; \quad b \leq x < c \\ 1 - \epsilon & ; \quad x \geq c \end{cases}$$

The diagrammatic representation of membership and non-membership functions are shown in Figure 2.1.

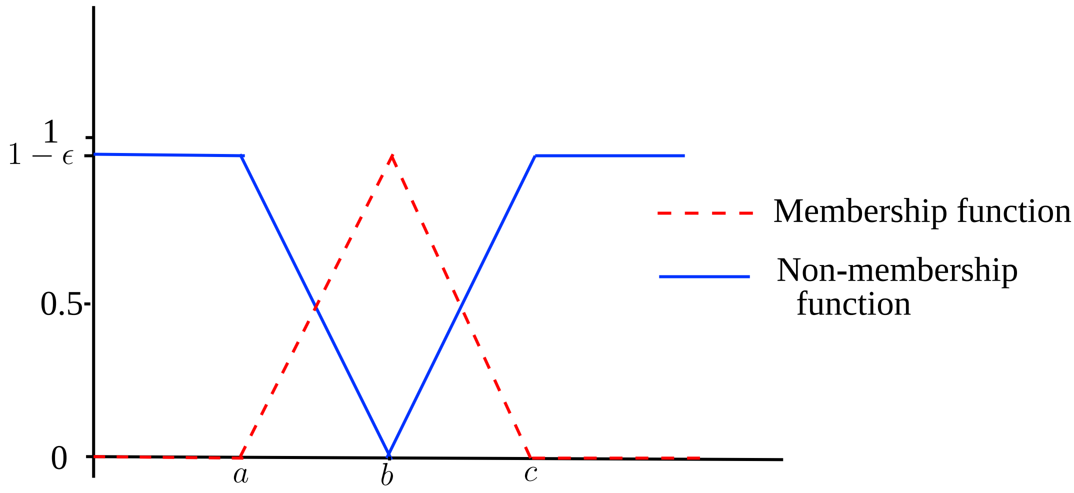


Figure 2.1: Intuitionistic fuzzy triangular function

Note

When $\epsilon = 0$, *iftrif* tends to *trif* in fuzzy.

Note

Hereafter, ϵ is an arbitrary parameter chosen in such a way that $\mu_A(x)+\nu_A(x)+\epsilon = 1$ and $0 < \epsilon < 1$.

Example 2.3.1. Suppose the room temperature varies from $-5°C$ to $+5°C$, then the corresponding membership and non-membership triangular functions for *approximately zero degree celsius* temperature specified by the three parameters $a = -5$, $b = 0$ and $c = +5$ are as follows: $(\epsilon = 0.1)$

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq -5 \\ (\frac{x+5}{5}) - 0.1 & ; \quad -5 < x \leq 0 \\ (\frac{5-x}{5}) - 0.1 & ; \quad 0 \leq x < 5 \\ 0 & ; \quad x \geq 5 \end{cases}$$

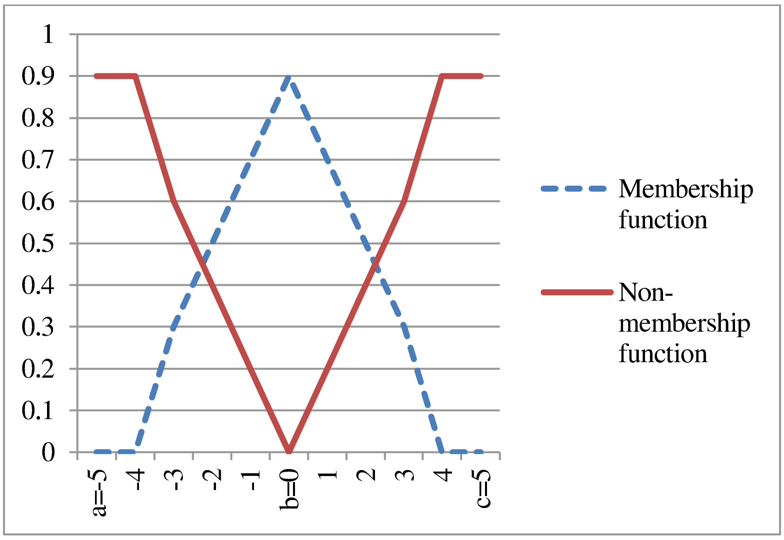


Figure 2.2: Intuitionistic fuzzy triangular function

$$\nu_A(x) = \begin{cases} 0.9 & ; \quad x \leq -5 \\ 1 - (\frac{x+5}{5}) & ; \quad -5 < x \leq 0 \\ 1 - (\frac{5-x}{5}) & ; \quad 0 < x \leq 5 \\ 0.9 & ; \quad x \geq 5 \end{cases}$$

The $iftrif$ for the intuitionistic fuzzy set *approximately zero degree celsius* is shown in Figure 2.2.

2.3.2 Intuitionistic fuzzy trapezoidal function (*iftraf*)

Intuitionistic fuzzy trapezoidal function (*iftraf*), has a flat top and is a truncated triangle. The *iftraf* function is defined by four parameters, a lower limit a, an upper limit d, a lower support limit b and an upper support limit c, where $a \leq b \leq c \leq d$. Here, a and d locate the *feet* of the trapezium and b and c locate the *shoulder* point. The intuitionistic fuzzy trapezoidal membership function is defined as follows:

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq a \\ (\frac{x-a}{b-a}) - \epsilon & ; \quad a < x < b \\ 1-\epsilon & ; \quad b \leq x \leq c \\ (\frac{d-x}{d-c}) - \epsilon & ; \quad c < x < d \\ 0 & ; \quad x \geq d \end{cases}$$

The corresponding intuitionistic fuzzy trapezoidal non-membership function is given by

$$\nu_A(x) = \begin{cases} 1-\epsilon & ; \quad x \leq a \\ 1-(\frac{x-a}{b-a}) & ; \quad a < x < b \\ 0 & ; \quad b \leq x \leq c \\ 1-(\frac{d-x}{d-c}) & ; \quad c < x < d \\ 1-\epsilon & ; \quad x \geq d \end{cases}$$

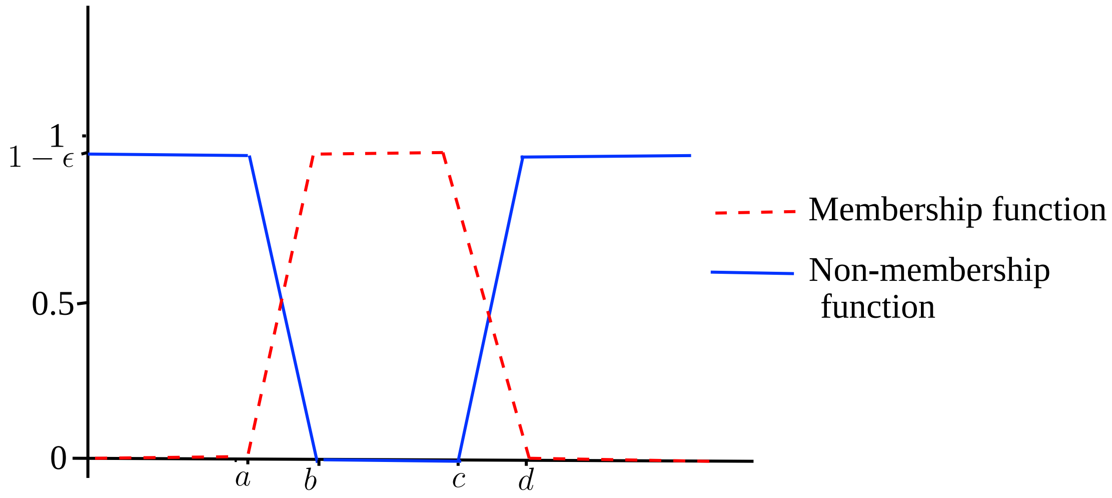


Figure 2.3: Intuitionistic fuzzy trapezoidal function

The graph of the intuitionistic fuzzy trapezoidal functions is displayed in Figure 2.3. The intuitionistic fuzzy trapezoidal functions may be symmetric or asymmetric in shape. The symmetric *iftraf* function is shown in Figure 2.3. Obviously, the *intuitionistic fuzzy triangular function* is a special case of *intuitionistic fuzzy trapezoidal function*.

Example 2.3.2. In problems like testing the youthness of the people according to the age of a person, the trapezoidal membership function may be used. Suppose A be the set of ages of *old men* which vary *around 55*. Assuming that men whose ages above 65 is treated as *very old*. In this example, the trapezoidal membership function is specified by the parameters $\{a = 50,\, b = 55,\, c = 60$ and $d{=}65\}$ and the corresponding membership and non-membership functions are defined as follows: $(\epsilon = 0.2)$

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq 50 \\ (\frac{x-50}{5}) - 0.2 & ; \quad 50 < x < 55 \\ 0.8 & ; \quad 55 \leq x \leq 60 \\ (\frac{65-x}{5}) - 0.2 & ; \quad 60 < x < 65 \\ 0 & ; \quad x \geq 65 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.8 & ; \quad x \leq 50 \\ 1 - (\frac{x-50}{5}) & ; \quad 50 < x < 55 \\ 0 & ; \quad 55 \leq x \leq 60 \\ 1 - \left(\frac{65-x}{5}\right) & ; \quad 60 < x < 65 \\ 0.8 & ; \quad x \geq 65 \end{cases}$$

The intuitionistic fuzzy trapezoidal functions, are categorized into two, namely, *intuitionistic fuzzy R-functions* and *intuitionistic fuzzy L-functions.*

2.3.3 Intuitionistic fuzzy R-functions

An *intuitionistic fuzzy R-function* is the right intuitionistic fuzzy *trapezoidal* function. *Intuitionistic fuzzy R-function* is specified by two parameters c and d with $a = b = -\infty$, whose membership functions is defined as follows:

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \geq d \\ (\frac{d-x}{d-c}) - \epsilon & ; \quad c < x < d \\ 1 - \epsilon & ; \quad x \leq c \end{cases}$$

The corresponding non-membership function takes the form

$$\nu_A(x) = \begin{cases} 1 - \epsilon & ; \quad x \geq d \\ 1 - (\frac{d-x}{d-c}) & ; \quad c < x < d \\ 0 & ; \quad x \geq c \end{cases}$$

The diagrammatic representation of intuitionistic fuzzy R-function is exhibited in Figure 2.4.

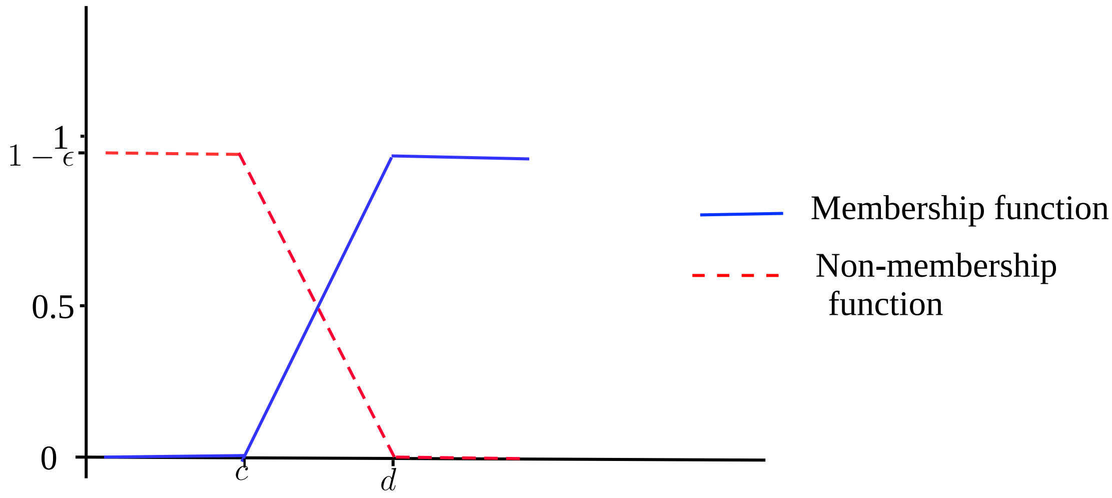


Figure 2.4: Intuitionistic fuzzy R-function

Example 2.3.3. If the parameters of the *intuitionistic fuzzy R -function* are specified by the parameters $c = 5.6$, $d = 5.8$, then the corresponding membership and non-membership functions are as follows : $(\epsilon = 0.2)$

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \geq 5.8 \\ (\frac{5.8-x}{0.2}) - 0.2 & ; \quad 5.6 < x < 5.8 \\ 0.8 & ; \quad x \leq 5.6 \end{cases}$$

$$\nu_A(x) = \begin{cases} 0.8 & ; \quad x \geq 5.8 \\ 1 - (\frac{5.8-x}{0.2}) & ; \quad 5.6 < x < 5.8 \\ 0 & ; \quad x \geq 5.6 \end{cases}$$

2.3.4 Intuitionistic fuzzy L-functions

Intuitionistic fuzzy L-function is the left intuitionistic fuzzy trapezoidal function. *Intuitionistic fuzzy L-* function is specified by two parameters a and b with

$c = d = +\infty$, whose membership takes the form

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq a \\ (\frac{x-a}{b-a}) - \epsilon & ; \quad a < x < b \\ 1 - \epsilon & ; \quad x \geq b \end{cases}$$

The corresponding, non-membership function is given as

$$\nu_A(x) = \begin{cases} 1 - \epsilon & ; \quad x \leq a \\ 1 - (\frac{x-a}{b-a}) & ; \quad a < x < b \\ 0 & ; \quad x \geq b \end{cases}$$

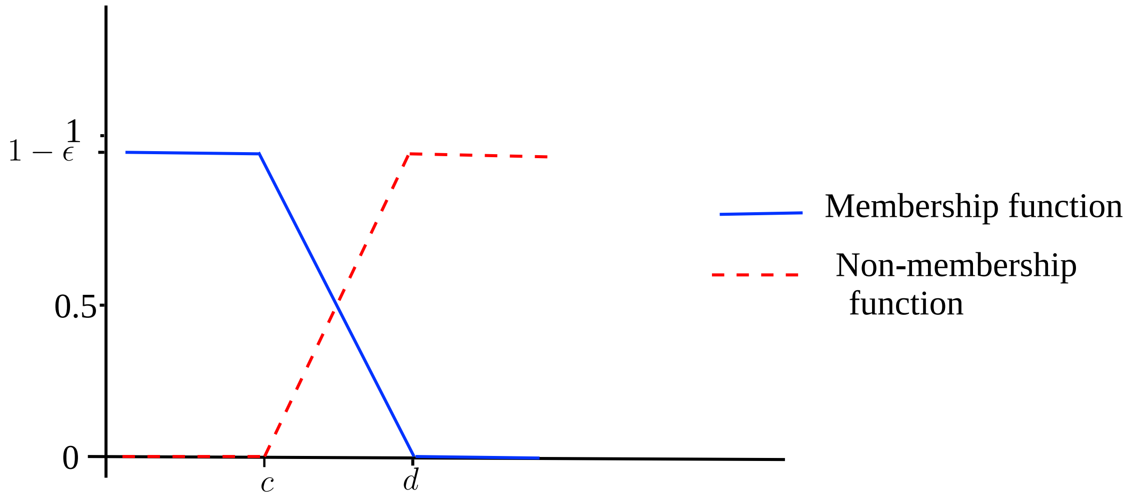


Figure 2.5: Intuitionistic fuzzy L-function

The diagrammatic representation of intuitionistic fuzzy L-function is presented in Figure 2.5.

2.3.5 Intuitionistic fuzzy Gaussian function (*ifgaussf*)

Intuitionistic fuzzy Gaussian function (*Ifgaussf*) is specified by two parameters. The Gaussian function is defined by a central value m and width $k > 0$. The smaller the k, the narrower the curve is. Intuitionistic fuzzy Gaussian membership and non-membership functions are defined as

$$\mu_A(x) = \exp(-\frac{(x-m)^2}{2(k)^2}) - \epsilon$$

$$\nu_A(x) = 1 - \left(\exp(-\frac{(x-m)^2}{2(k)^2})\right)$$

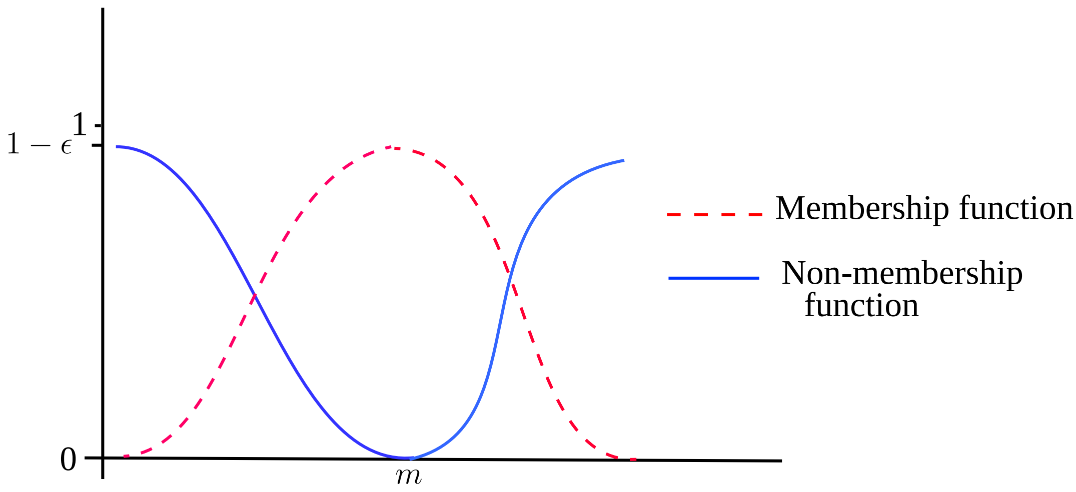


Figure 2.6: Intuitionistic fuzzy Gaussian function

The diagrammatic representation of intuitionistic fuzzy Gaussian function is shown in Figure 2.6.

Example 2.3.4. The exponential growth of the bacteria can be expressed by the intuitionistic fuzzy Gaussian function. If the Gaussian membership function is determined by the parameters $m = 5$ and $k = 1$, then the Gaussian membership and non-membership functions are as follows ($\epsilon = 0.01$):

$$\mu_A(x) = \exp(-\tfrac{(x-5)^2}{2}) - 0.01$$

and

$$\nu_A(x) = 1 - \left(\exp(-\tfrac{(x-5)^2}{2})\right)$$

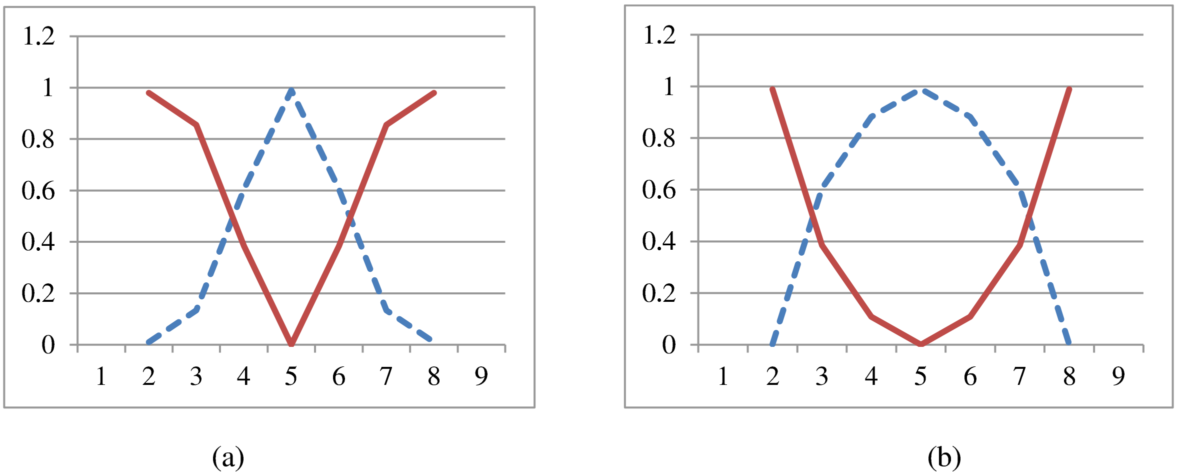


Figure 2.7: Intuitionistic fuzzy gaussian function (a) $m = 5$, $k = 1$ (b) $m = 5$, $k = 0.5$

The diagrammatic representation of intuitionistic fuzzy Gaussian function for $m = 5$, $k = 1$ and $m = 5$, $k = 0.5$ is shown in Figure 2.7(a) and 2.7(b) respectively.

2.3.6 Intuitionistic fuzzy bell-shaped function (*ifbellf*)

Intuitionistic fuzzy bell-shaped function is specified by three parameters a, b, c and usally the parameter b is positive. The parameter c locates the center of the curve and b control the slopes at the crossover points. The intuitionistic fuzzy bell-shaped membership and non-membership functions are defined as

$$\mu_A(x) = 1 - \epsilon - \left(\frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}\right)$$

and

$$\nu_A(x) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

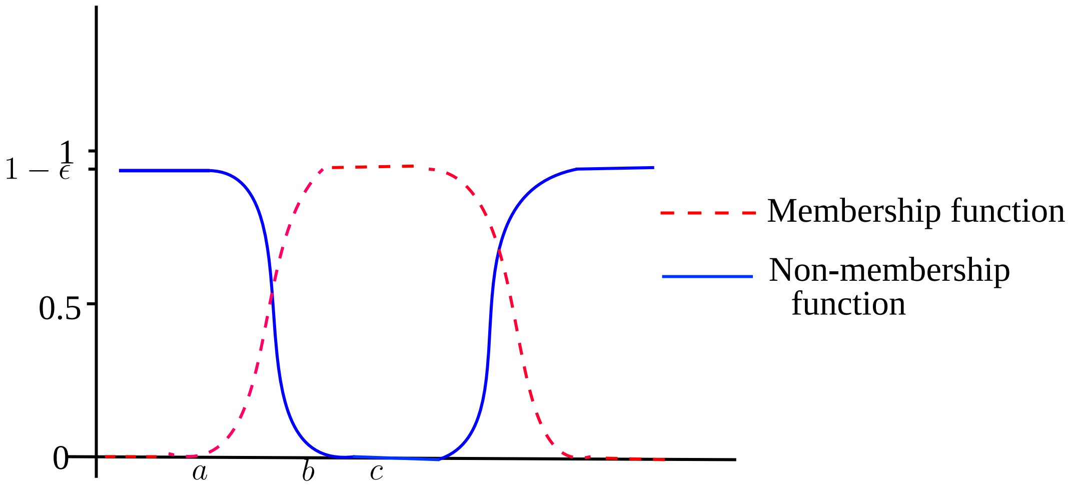


Figure 2.8: Intuitionistic fuzzy bell-shaped function

As the shape of the membership resembles the bell and that of non-membership resembles the inverted bell in Figure 2.8, the function is called intuitionistic fuzzy bell-shaped function.

2.3.7 Intuitionistic fuzzy sigmoidal function (*ifsigf*)

Intuitionistic fuzzy sigmoidal function depends on two parameters a and c, where c locates the distance from the origin and a determines the steepness of the function. Depending on the sign of the parameter a, the intuitionistic fuzzy sigmoidal membership function is inherently open to the right or to the left. If a is positive, the function is open to the right, whereas if it is negative it is open to the left. As the parameter increases, the transition from 0 to 1 becomes sharper.

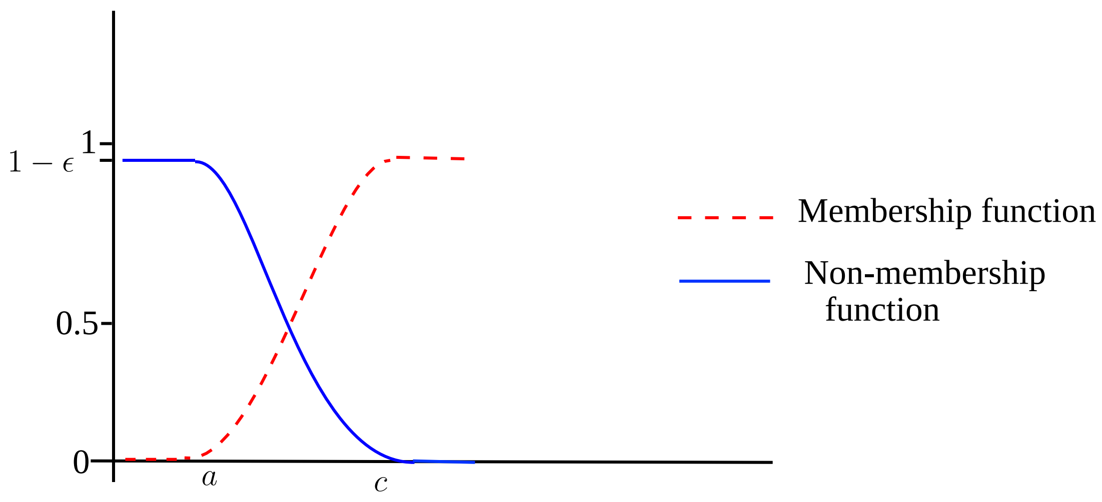


Figure 2.9: Intuitionistic fuzzy Sigmoidal function

The intuitionistic fuzzy sigmoidal membership and non-membership functions are defined as

$$\mu_A(x) = \left(\frac{1}{1+\exp(-a(x-c))}\right) - \epsilon$$

and

$$\nu_A(x) = 1 - \frac{1}{1+\exp(-a(x-c))}$$

In Figure 2.9, the intuitionistic fuzzy sigmoidal function is open to the right. Intuitionistic fuzzy sigmoidal function is commonly used as an activation function in neural networks.

2.3.8 Intuitionistic fuzzy S-shaped function (*ifSf*)

The precise appearance of *ifSf* is determined by the choice of the parameters a, b and the parameters locate the extremes of the sloped portion of the curve. Intuitionistic fuzzy S-shaped membership function takes the form

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq a \\ 2(\frac{x-a}{b-a})^2 - \epsilon & ; \quad a < x \leq \frac{a+b}{2} \\ 1 - 2(\frac{x-b}{b-a})^2 - \epsilon & ; \quad \frac{a+b}{2} \leq x < b \\ 1 - \epsilon & ; \quad x \geq b \end{cases}$$

Similarly, intuitionistic fuzzy S-shaped non-membership function is given by

$$\nu_A(x) = \begin{cases} 1 - \epsilon & ; \quad x \leq a \\ 1 - 2(\frac{x-a}{b-a})^2 & ; \quad a < x \leq \frac{a+b}{2} \\ 2(\frac{x-b}{b-a})^2 & ; \quad \frac{a+b}{2} \leq x < b \\ 0 & ; \quad x \geq b \end{cases}$$

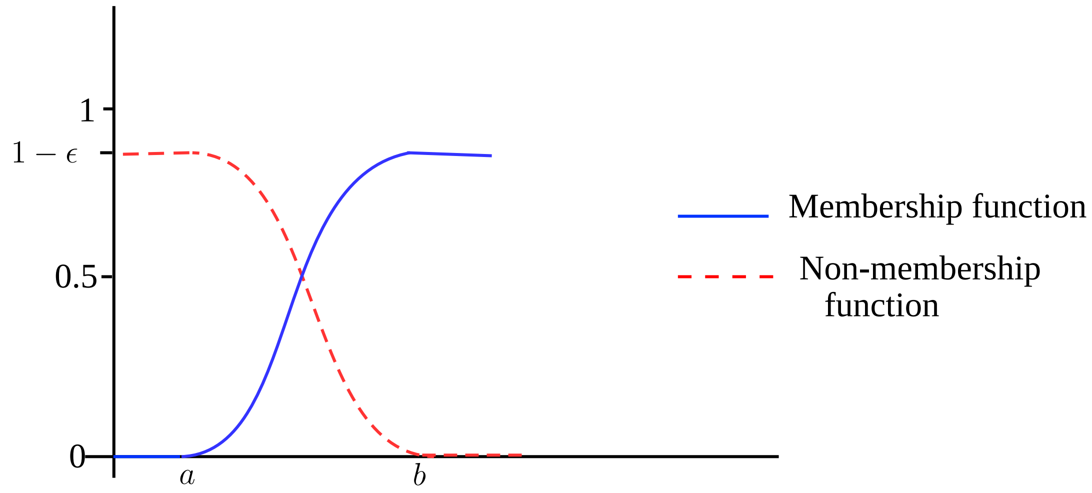


Figure 2.10: Intuitionistic fuzzy S-shaped function

The graphical representation of intuitionistic fuzzy S-shaped function is depicted in Figure 2.10.

Example 2.3.5. If the two parameters of the intuitionistic fuzzy S-shaped function are given to be $a = 5.1$ and $b = 5.5$, then the corresponding membership and non-membership functions are as follows ($\epsilon = 0.1$):

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq 5.1 \\ 2(\frac{x-5.1}{0.4})^2 - 0.1 & ; \quad 5.1 < x \leq 5.3 \\ 1 - 2(\frac{x-5.5}{0.4})^2 - 0.1 & ; \quad 5.3 \leq x < 5.5 \\ 0.9 & ; \quad x \geq 5.5 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.9 & ; \quad x \leq 5.1 \\ 1 - 2(\frac{x-5.1}{0.4})^2 & ; \quad 5.1 < x \leq 5.3 \\ 2(\frac{x-5.5}{0.4})^2 & ; \quad 5.3 \leq x < 5.5 \\ 0 & ; \quad x \geq 5.5 \end{cases}$$

2.3.9 Intuitionistic fuzzy Z-shaped function (*ifZf*)

The *ifZf*, is given by two parameters, a and b which locate the extremes of the sloped portion of the curve.

Intuitionistic fuzzy Z-shaped membership function is defined as

$$\mu_A(x) = \begin{cases} 1-\epsilon & ; \quad x \leq a \\ 1-2(\frac{x-a}{b-a})^2-\epsilon & ; \quad a < x \leq \frac{a+b}{2} \\ 2(\frac{x-b}{b-a})^2-\epsilon & ; \quad \frac{a+b}{2} \leq x < b \\ 0 & ; \quad x \geq b \end{cases}$$

The corresponding intuitionistic fuzzy Z-shaped non-membership function takes the form

$$\nu_A(x) = \begin{cases} 0 & ; \quad x \leq a \\ 2(\frac{x-a}{b-a})^2 & ; \quad a < x < \frac{a+b}{2} \\ 1-2(\frac{x-a}{b-a})^2 & ; \quad \frac{a+b}{2} \leq x < b \\ 1-\epsilon & ; \quad x \geq b \end{cases}$$

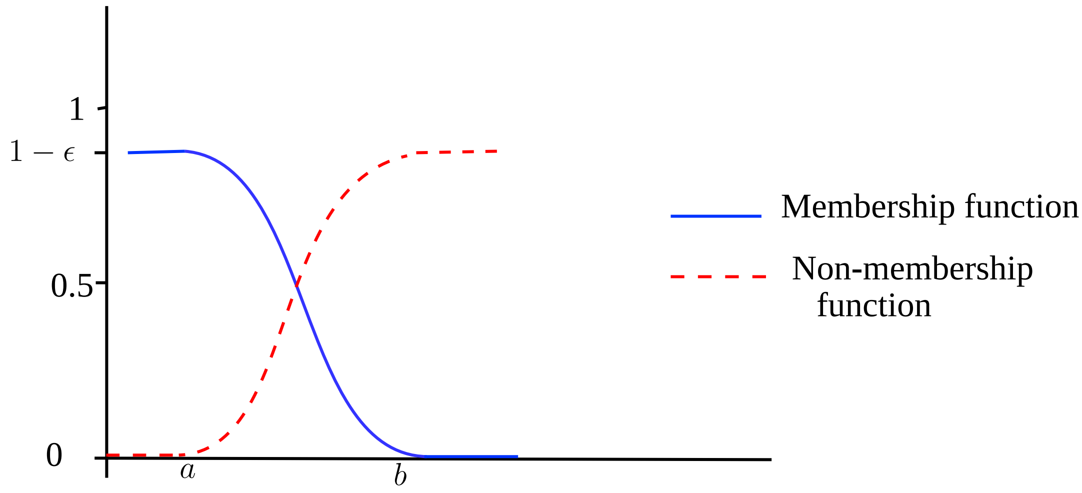


Figure 2.11: Intuitionistic fuzzy Z-shaped function

The diagrammatic representation of intuitionistic fuzzy Z-shaped function is shown in Figure 2.11.

Example 2.3.6. If the two parameters of the intuitionistic fuzzy Z-shaped function are given to be $a = 5.1$ and $b = 5.5$, then the corresponding membership and non-membership functions are as follows ($\epsilon = 0.1$):

$$\mu_A(x) = \begin{cases} 0.9 & ; \quad x \leq 5.1 \\ 1 - 2(\frac{x-5.1}{0.4})^2 - 0.1 & ; \quad 5.1 < x \leq 5.5 \\ 2(\frac{x-5.5}{0.4})^2 - 0.1 & ; \quad 5.3 \leq x < 5.5 \\ 0 & ; \quad x \geq 5.5 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & ; \quad x \leq 5.1 \\ 2(\frac{x-5.1}{0.4})^2 & ; \quad 5.1 < x \leq 5.3 \\ 1 - 2(\frac{x-5.5}{0.4})^2 & ; \quad 5.3 \leq x < 5.5 \\ 0.9 & ; \quad x \geq 5.5 \end{cases}$$

2.4 Defuzzification of intuitionistic fuzzy sets

Defuzzification is the process of converting a fuzzy quantity to precise quantity, just as fuzzification is the conversion of a precise quantity to a fuzzy quantity. As output of any mathematical model must be crisp, it is necessary to convert the modified intuitionistic fuzzy values into crisp, which is termed as *IF-defuzzification*. Various types of defuzzification methods are available for converting fuzzy to non-fuzzy. In this section, intuitionistic defuzzification functions such as triangular, trapezoidal, L-trapezoidal, R-trapezoidal, gaussian, S-shaped, Z-shaped functions are defined.

Definition 2.4.1. [9]

Let the universal set X be fixed and A be an IFS, then modal operator $\boxplus$ is defined as

$$\boxplus A = \left\{ \langle x, \tfrac{\mu_A(x)}{2}, \tfrac{\gamma_A(x)+1}{2} \rangle : x \in X \right\}$$

Definition 2.4.2. [9]

Let the universal set X be fixed and A be an IFS, then modal operator $\boxtimes$ is defined as

$$\boxtimes A = \left\{ \langle x, \tfrac{\mu_A(x)+1}{2}, \tfrac{\gamma_A(x)}{2} \rangle : x \in X \right\}$$

2.4.1 Intuitionistic fuzzy-defuzzification functions

hspace0.5cm *Intuitionistic fuzzy (IF)-defuzzification functions* are used to convert membership and non-membership values into precise quantity. The term

IF-defuzzification function (IFDF) refers to defuzzification function of an IFS. This section defines and discusses few IF defuzzification functions and suitable examples.

2.4.2 Intuitionistic fuzzy triangular defuzzification function (*iftridf*)

IF-triangular defuzzification function (iftridf) is given by,

$$C(y) = \begin{cases} \leq a & \text{if} \quad y = 0 \\ a + (b-a)(y+\epsilon) - (\sqrt{\mu * (c_1 - \nu)}) & \text{if} \quad 0 < y \leq \frac{x-a}{b-a} - \epsilon \\ (b-a)(y+\epsilon) + c - \sqrt{\mu * (c_2 - \nu)} & \text{if} \quad \frac{x-a}{b-a} - \epsilon \leq y < \frac{c-x}{c-b} - \epsilon \\ \geq c & \text{if} \quad y = 0 \end{cases}$$

where c_1 and c_2 are arbitrary constants and $y = \mu_A(x)$ is the fuzzified value which lies in $[0, 1]$ and ϵ is a small quantity such that $\mu_A(x) + \nu_A(x) + \epsilon = 1$ and $0 < \epsilon < 1$.

Note

Hereafter, ϵ is a small quantity chosen in such a way that $\mu_A(x) + \nu_A(x) + \epsilon = 1$ and $0 < \epsilon < 1$.

2.4.3 Intuitionistic fuzzy trapezoidal defuzzification function (*iftradf*)

IF-trapezoidal defuzzification function (iftradf) is given by

$$C(y) = \begin{cases} \leq a & \text{if } \; y = 0 \\ a + (b-a)(y+\epsilon) - (\sqrt{\mu * (c_1 - \nu)}) & \text{if } \; 0 < y \leq \frac{x-a}{b-a} - \epsilon \\ b \leq x \leq c & \text{if } \; y = 1 - \epsilon \\ (c-d)(y+\epsilon) + d - \sqrt{\mu * (c_2 - \nu)} & \text{if } \; 1 - \epsilon < y < \frac{d-x}{d-c} - \epsilon \\ \geq d & \text{if } \; y = 0 \end{cases}$$

where c_1 and c_2 are arbitrary constants.

Intuitionistic fuzzy R-trapezoidal defuzzification function (*ifrtdf*)

The IF-R-trapezoidal defuzzification function (ifrtdf) is given by

$$C(y) = \begin{cases} \leq c & \text{if } \; y = 1 - \epsilon \\ (c-d)(y+\epsilon) + d - \sqrt{\mu * (c_1 - \nu)} & \text{if } \; 1 - \epsilon < y < \frac{d-x}{d-c} - \epsilon \\ \geq d & \text{if } \; y = 0 \end{cases}$$

where c_1 is an arbitrary constant.

Intuitionistic fuzzy L-trapezoidal defuzzification function (*ifltdf*)

IF-L-trapezoidal defuzzification function (ifltdf) takes the form

$$C(y) = \begin{cases} \leq a & \text{if} \quad y = 0 \\ a + (b-a)(y+\epsilon) - \sqrt{\mu * (c_2 - \nu)} & \text{if} \quad 0 < y \leq \frac{x-a}{b-a} - \epsilon \\ \geq b & \text{if} \quad y \geq 1 - \epsilon \end{cases}$$

where c_2 is an arbitrary constant.

2.4.4 Intuitionistic fuzzy Gaussian defuzzification functions (*ifgaussdf*)

IF Gaussian defuzzification functions (ifgaussdf) are defined as

$$C(y) = \begin{cases} m - k\sqrt{2(log(y+\epsilon))} - \sqrt{\mu * c_1 * \gamma} & \text{if} \quad x \leq m \\ m + k\sqrt{2(log(y+\epsilon))} + \sqrt{\mu * c_2 * \gamma} & \text{if} \quad x > m \end{cases}$$

where c_1 and c_2 are arbitrary constants.

2.4.5 Intuitionistic fuzzy S-shaped defuzzification function (*ifSdf*)

IF-S-shaped defuzzification function (ifSdf) takes the form

$$C(y) = \begin{cases} \leq a & \text{if} \quad y = 0 \\ a + \frac{(b-a)\sqrt{y+\epsilon}}{\sqrt{2}} + (\mu * (c_1 - \nu))^2 & \text{if} \quad 0 < y \leq 2(\frac{x-a}{b-a})^2 - \epsilon \\ b - \frac{(b-a)\sqrt{1-(y+\epsilon)}}{\sqrt{2}} - (\mu * c_2 * \nu)^2 & \text{if} \quad 2(\frac{x-a}{b-a})^2 - \epsilon \leq y < 1 - 2(\frac{x-b}{b-a})^2 - \epsilon \\ \geq b & \text{if} \quad y \geq 1 - \epsilon \end{cases}$$

where c_1 and c_2 are arbitrary constants.

2.4.6 Intuitionistic fuzzy Z-shaped defuzzification function (*ifZdf*)

IF-Z-shaped defuzzification function (ifZdf) is defined as

$$C(y) = \begin{cases} \leq a & \text{if} \quad y = 1 - \epsilon \\ a + \frac{(b-a)\sqrt{1-(y+\epsilon)}}{\sqrt{2}} + (\mu * c_1 * \nu)^2 & \text{if} \quad 0 < y \leq 1 - 2(\frac{x-a}{b-a})^2 - \epsilon \\ b - \frac{(b-a)\sqrt{y+\epsilon}}{\sqrt{2}} - (\mu * (c_2 - \nu))^2 & \text{if} \quad 1 - 2(\frac{x-a}{b-a})^2 - \epsilon \leq y < 2(\frac{x-b}{b-a})^2 - \epsilon \\ \geq b & \text{if} \quad y = 0 \end{cases}$$

where c_1 and c_2 are arbitrary constants.

2.5 Numerical Examples

Example 2.5.1. In this example, IF-triangular fuzzification function is used for fuzzification and *iftridf* is used for defuzzification processes. The other defuzzification functions can also be verified in a similar way. Consider a 3×3 grey matrix extracted from an image whose grey values vary from 0 to 255. The parameters are $a = 0$, $b = 128$, $c = 256$ and $\epsilon = 0.001$. This example is done only to check the validity of the proposed intuitionistic defuzzification functions.

$$A = \begin{bmatrix} 50 & 128 & 192 \\ 202 & 220 & 166 \\ 256 & 32 & 64 \end{bmatrix} \tag{2.1}$$

From Definition 2.3.1 of *iftrif*, IF-triangular fuzzified 3×3 matrix is given by

$$[\langle \mu_A(x), \nu_A(x) \rangle] = \begin{bmatrix} \langle 0.3896, 0.6094 \rangle & \langle 0.9990, 0 \rangle & \langle 0.4990, 0.5000 \rangle \\ \langle 0.4206, 0.5781 \rangle & \langle 0.2803, 0.7188 \rangle & \langle 0.7021, 0.2969 \rangle \\ \langle 0, 0, 9990 \rangle & \langle 0.2490, 0.7500 \rangle & \langle 0.4990, 0.5000 \rangle \end{bmatrix} \tag{2.2}$$

The corresponding IF-triangular defuzzified matrix is obtained as follows

$$A = \begin{bmatrix} 49.6099 & 127.0005 & 191.1348 \\ 201.2231 & 219.7128 & 164.9096 \\ 256 & 31.7505 & 63.5005 \end{bmatrix} \tag{2.3}$$

Here, no modification is carried out in the input. After IF triangular fuzzification (iftrif) and IF triangular defuzzification process (iftridf) the output remains the same. Comparing (2.1) and (2.3), the loss of accuracy is mainly due to numerical approximation.

Example 2.5.2. Intuitionistic fuzzification, modification of membership and non-membership values and intuitionistic defuzzification are the three major steps involved in modeling real situations via IFSs. IF logic controller models human experience, human decision making behaviour and so on. In intuitionistic fuzzy inference system, modification is required so that result is more suitable than original for perception. In this example, intuitionistic fuzzy modal operators $\boxplus$ and $\boxtimes$ are used for modification of membership and non-membership values. Consider the same matrix A as in Example 2.5.1.
Based on the matrix (2.2), the modified matrix using modal operator $\boxplus$, is given by

$$\left[\left\langle \mu_A^{'}(x), \nu_A^{'}(x) \right\rangle\right]_{3\times 3} = \begin{bmatrix} \langle 0.1948, 0.8047\rangle & \langle 0.4995, 0.5\rangle & \langle 0.2495, 0.75\rangle \\ \langle 0.2103, 0.7890\rangle & \langle 0.1401, 0.8594\rangle & \langle 0.3510, 0.6484\rangle \\ \langle 0, 0.9995\rangle & \langle 0.1245, 0.875\rangle & \langle 0.2495, 0.75\rangle \end{bmatrix} \tag{2.4}$$

The corresponding defuzzified matrix for (2.4) is as follows

$$C = \begin{bmatrix} 24.8673 & 63.5642 & 223.3775 \\ 228.4489 & 237.5394 & 210.2552 \\ 255.872 & 15.9392 & 31.8142 \end{bmatrix} \tag{2.5}$$

The modified matrix using modal operator $\boxtimes$, is given by

$$\left[\left\langle \mu''_A(x), \nu''_A(x) \right\rangle\right]_{3\times 3} = \begin{bmatrix} \langle 0.6948, 0.3047\rangle & \langle 0.9995, 0\rangle & \langle 0.7495, 0.25\rangle \\ \langle 0.7103, 0.2890\rangle & \langle 0.6401, 0.3594\rangle & \langle 0.8510, 0.1484\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.6245, 0.375\rangle & \langle 0.7495, 0.25\rangle \end{bmatrix} \tag{2.6}$$

The corresponding defuzzified matrix for (2.6) is as follows:

$$C = \begin{bmatrix} 88.3673 & 127.0642 & 159.1862 \\ 90.3357 & 172.9144 & 145.6887 \\ 191.005 & 79.2887 & 95.3142 \end{bmatrix} \tag{2.7}$$

Comparing (2.5) and (2.7), it is inferred after applying $\boxplus$ modal operator, that they can be used for enhancement. Moreover, low intensity grey levels are further decrease and high intensity gray levels are further increased. The modal operator $\boxtimes$ increases the low and decreases the high. These tools find applications in image processing.

Chapter 3

Intuitionistic fuzzy logic controller

3.1 Introduction

In this chapter, an architecture of intuitionistic fuzzy logic controller (IFLC) is designed so that it can be used for any controlling system. As designing of a common IFLC is not found anywhere in the literature, it is therefore necessary to design IFLC as a tool. Further, new intuitionistic fuzzy operators are defined and applied in intuitionistic fuzzy inference engine to establish a flexible mathematical framework to model the vagueness, which will find applications in noise removal in image processing. Finally, the validity of the controller is verified with suitable illustrations.

3.2 Architecture of an intuitionistic fuzzy logic controller

The basic structure of an IFLC is shown in Figure 3.1. The controller includes the following components.

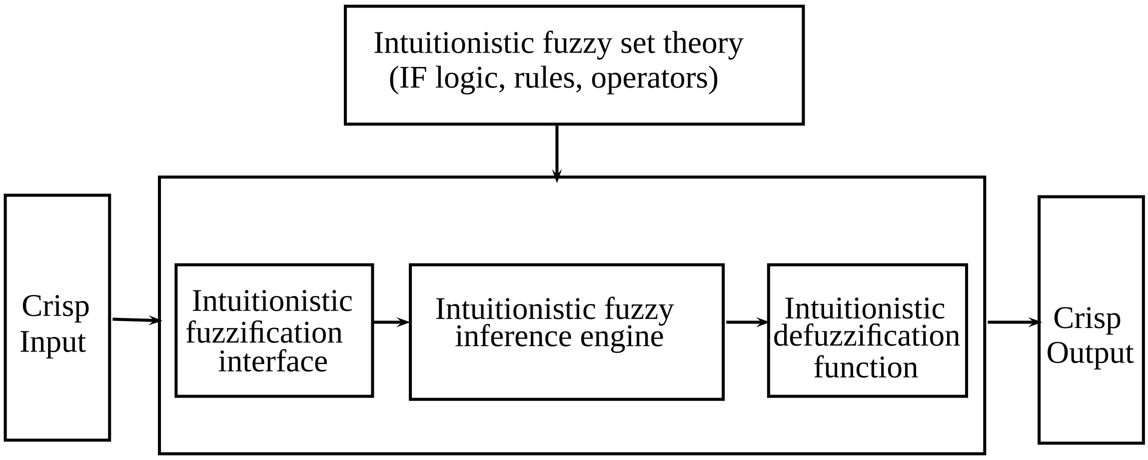


Figure 3.1: Block diagram of an intuitionistic fuzzy logic controller

1. **Intuitionistic fuzzifier**

   Intuitionistic fuzzification transforms input crisp values into intuitionistic fuzzy values. Nine types of intutionistic fuzzification functions are defined on the basis of different shapes of the membership and non-membership functions. Appropriate intuitionistic fuzzy membership function can be selected for intuitionistic fuzzification.

2. **Intuitionistic fuzzy inference engine**

   Intuitionistic fuzzy inference engine is composed of intuitionistic fuzzy IF-THEN rules, intuitionistic fuzzy logic, intuitionistic fuzzy operators etc. These elements are used for modification of intuitionistic fuzzy pairs as required.

3. **Intuitionistic defuzzifier**

   As output of any mathematical model must be crisp, it is applicable for IFLC also. Hence, it is necessary to convert the modified intuitionistic fuzzy values into crisp, which is termed as intutionistic defuzzification. Various types of intutionistic defuzzification functions such as triangular, trapezoidal, L-trapezoidal, R- trapezoidal, S-shaped, Z-shaped functions are defined. Suitable intuitionistic fuzzy defuzzification function for a specified need can be selected for defuzzification.

Remarks

1. If any of the defined intuitionistic fuzzification functions is not found suitable for the specific problem, the user can define a required function, based on the requirement.

2. Unlike defuzzification to single value in fuzzy controller, intuitionistic defuzzification gives a matrix of defuzzified values in [0,1], which is more suitable for designing IFLC in image processing.

3.3 An example in image processing

Image Processing is a form of information processing for which the input is an image, such as a photograph or video frame, the output may be either an image or a set of characteristics related to the image where, an image is defined as an array, or a matrix, square pixel arranged in rows and columns [84]. Most image processing techniques involve treating the image as a two-dimensional representation and applying standard processing techniques to it.

Noise is an unwanted effect in an image which degrades the image to different extend during image acquistion or transmission [84]. A *noisy image* can be modeled as

$$C(X, Y) = A(X, Y) + B(X, Y)$$

where $A(X, Y)$ is the original image, $B(X, Y)$ is the noise in the image and $C(X, Y)$ is the resulting noisy image. Common types of noise in images are *impulse noise* and *random noise*. Salt and pepper noise is a special case of impulse noise.

Consider a greyscale image $'x'$ defined as $m \times n$ matrix , where $x(i, j)$ represents the intensity of the pixel at the i^{th} row and the j^{th} column. The intensity is stored in an 8-bit integer, giving 256 possible grey levels in the interval [0, 255]. In this interval, a salt and pepper noise takes minimum and maximum intensity and appears in digital image with equal probabilities. The noise can be positive or negative. Positive impulse appears as white (salt) points with intensity 255 and negative impulse appears as black (pepper) points with intensity 0.

Notations

Let $A = (A_1, A_2, \cdots A_n)$ be a sample of n intuitionistic fuzzy data, shortly IFSp -A, let E be a fixed non-empty set. Let $\langle \mu_{A_i}(x), \nu_{A_i}(x) \rangle, i = 1, 2, \cdots n$ be the membership and non-membership values of x in A.

Definition 3.3.1.

The *mean* of IFSp -A, denoted by $IF_{\bar{A}}$, is defined as

$$IF_{\bar{A}} = \left\{ \left\langle x, \frac{\sum_{i=1}^{n} \mu_{A_i}(x)}{n}, \frac{\sum_{i=1}^{n} \nu_{A_i}(x)}{n} \right\rangle : x \in E \right\}$$

Definition 3.3.2.

The *median* of an IFSp -A, denoted by $IF_{med}(A)$, defined as

$$IF_{med}(A) = \{\langle x, med(\mu_{A_i}(x), med(\nu_{A_i}(x))\rangle : x \in E\}$$

Definition 3.3.3. [75]

Let A be an IFS in E, then its *Normalization* denoted by $Nor(A)$, is defined as

$$Nor(A) = \Big\{\Big\langle x, \mu_{Nor(A)}(x), \nu_{Nor(A)}(y)\Big\rangle : x \in E\Big\}$$

where $\mu_{Nor(A)}(x) = \frac{\mu_A(x)}{sup(\mu_A(x))}$ and $\nu_{Nor(A)}(x) = \frac{\nu_A(x) - inf(\nu_A(x))}{1 - inf(\nu_A(x))}$

Definition 3.3.4. [75]

The following are the steps to calculate the *mode* of an IFSp -A

Step 1. Normalize the data using

$$\mu_{Nor(A_i)}(x) = \frac{\mu_{A_i}(x)}{sup(\mu_{A_i}(x))}, \ \nu_{Nor(A_i)}(x) = \frac{\nu_{A_i}(x)}{sup(\nu_{A_i}(x))}$$

Step 2. Choosing minimum of normalized membership values and maximum of non-membership values for the IF mode denoted by $IF_{mode}(A)$.

Definition 3.3.5.

The *maximum* of an IFSp -A, denoted by $IF_{max}(A)$, defined as

$$IF_{max}(A) =$$
$$\{\langle x, max(\mu_{A_1}(x), \mu_{A_2}(x), \cdots \mu_{A_n}(x)), min(\nu_{A_1}(x), \nu_{A_2}(x), \cdots \nu_{A_n}(x))\rangle : x \in E\}$$

Definition 3.3.6.

The *minimum* of an IFSp -A, denoted by $IF_{min}(A)$, defined as

$IF_{min}(A) = \{\langle x, min(\mu_{A_1}(x), \mu_{A_2}(x), \cdots \mu_{A_n}(x)), max(\nu_{A_1}(x), \nu_{A_2}(x), \cdots \nu_{A_n}(x))\rangle : x \in E\}$

3.4 Proposed algorithm

Digital images are often corrupted by different kinds of noise due to errors that occur in the process of transmission in the communication channels and need to be processed to improve their pictorial information for better visual interpretation. In order to get a noise free image, several linear and non-linear filtering techniques are used [84].

Filters increase the brightness and contrast and add wide variety of special effects to an image. There are many types of filters and among them mean, median, mode play an important role in filtering. In recent years, many fuzzy filters have been designed to provide better results than traditional filters.

Information on grey level of images are always not of certain nature. In such situation, fuzzy logic is used to process imperfect data which arises due to vagueness and ambiquity. Inspite of vast applications, FSs are not always able to model uncertainties associated with imperfect information. This is due to the fact that their membership functions are themselves crisp. Data given as inputs to an fuzzy logic system and data used for tuning, are often noisy, thus bearing an amount of uncertainty.

IFSs provide an intuitive framework to deal vagueness from imprecise information by taking in to account non-membership values in addition to membership values. IFLC for noisy images is designed using intuitionistic fuzzy filters applied in inference engine and the proposed technique removes or effectively suppresses the noise in the image and enhances the image quality without any loss of edge information. Four different filtering techniques namely intuitionistic fuzzy (IF) mean, IF median, IF max, IF min filters are defined and their filtering performance on impulse noise is presented. The performance of the proposed method is evaluated in MATLAB simulations for an image that has been subjected to various degrees of corruption with impulse noise. The results demonstrate the effectiveness of the algorithm.

In this section, IF filtering algorithm is designed for noise reduction and image enhancement. The proposed technique removes the noise and improves image quality without any loss of information and following are the steps listed to develop the proposed model:

- Read the noisy image and obtain the grey level matrix.
- Choose $a =$ minimum grey level and $b =$ maximum grey level.
- Define the membership function

$$\mu_A(x) = \begin{cases} 0 & ; \quad x \leq a \\ 2(\frac{x-a}{b-a})^2 - \epsilon & ; \quad a < x \leq \frac{a+b}{2} \\ 1 - 2(\frac{x-b}{b-a})^2 - \epsilon & ; \quad \frac{a+b}{2} \leq x < b \\ 1 - \epsilon & ; \quad x \geq b \end{cases}$$

  where x is the grey level of the pixel which is to be fuzzified.

- Calculate the non-membership values in terms of membership values.

$$\nu_A(x) = \begin{cases} 1-\epsilon & ; \quad x \le a \\ 1-2(\frac{x-a}{b-a})^2 & ; \quad a < x \le \frac{a+b}{2} \\ 2(\frac{x-b}{b-a})^2 & ; \quad \frac{a+b}{2} \le x < b \\ 0 & ; \quad x \ge b \end{cases}$$

such that $0 \le \mu_A(x) + \nu_A(x) + \epsilon \le 1$.

- Modify membership and non-membership values $\langle \mu'_{mn}, \nu'_{mn} \rangle$ using any one of the intuitionistic fuzzy filters namely IF mean, IF maximum, IF minimum, IF median [using Definitions 3.2.1, 3.2.5, 3.2.6, 3.2.2] respectively.

- Calculate the new grey level using the modified membership and non-membership values.

$$g'_A = \begin{cases} a + \frac{(b-a)\sqrt{y+\epsilon}}{\sqrt{2}} + (\mu * (C_1 * \nu))^2 & ; \quad 0 \le y \le 2(\frac{x-a}{b-a})^2 - \epsilon \\ b - \frac{(b-a)\sqrt{y+\epsilon}}{\sqrt{2}} - (\mu * (C_2 * \nu))^2 & ; \quad 1-2(\frac{x-b}{b-a})^2 - \epsilon \le y < 2(\frac{x-a}{b-a})^2 - \epsilon \end{cases}$$

where C_1 and C_2 are arbitrary constants.

- Display noise-free output image.

3.5 Results and Discussion

Intuitionistic fuzzy statistical tools find better performance to handle imprecision in grey distribution. Experimental analysis is performed in two different ways. (i) Based on the image performance. (ii) Based on the statistical measures (Peak to signal noise ratio (PSNR)).

To illustrate, a grey scale lena image of size 256 ×256 with 8 bits per pixel tone resolution with 20% of salt and pepper noise is considered for analysis. IF filter of window size 3×3 is applied. grey value of the image, membership and non-membership values of the distored image and restored image are shown in Example 3.4.1, which interprets IF median filter perform better for elimination of salt and pepper noise. In the proposed system, non-membership values have also been included which makes difference in the output.

Example 3.5.1. For illustrative purpose, a particular part of the image is extracted and used for analysis.

Step(i): Original grey level of the given image are converted into double type.

0.623529	0.639215	0.643137
0.631372	0.639215	0.639215
0.639215	0.635294	0.635294
0.6431372	0.639215	0.631372
0.647058	0.639215	0.635294
0.631372	0.631372	0.627450
0.615686	0.623529	0.623529
0.615686	0.627450	0.631372
0.615686	0.619607	0
0.623529	0.623529	0.627450

Step(ii): Choose $a = 0$ and $b = 0.647058$.

Step(iii): The intuitionistically fuzzified values of the grey level are displayed as below:

Array-membership values **Array-non-membership values**

0.716439	0.739569	0.745197	0.283460	0.260330	0.254702
0.728127	0.739569	0.739569	0.271772	0.260330	0.260330
0.739569	0.733879	0.733879	0.260330	0.266020	0.266020
0.745197	0.739569	0.728127	0.254702	0.260330	0.271772
0.750765	0.739569	0.733879	0.249134	0.260330	0.266020
0.728127	0.728127	0.722314	0.271772	0.271772	0.277585
0.704505	0.716439	0.716439	0.295394	0.283460	0.283460
0.704505	0.722314	0.728127	0.295394	0.277585	0.271772
0.704505	0.710503	0	0.295394	0.289396	0.999900
0.716439	0.716439	0.722314	0.283460	0.283460	0.277585

Step(iv): The modified membership and non-membership values using IF median filter are calculated and tabulated as follows:

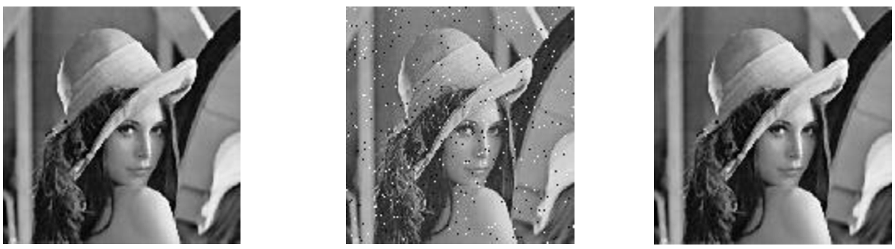
Modified membership values			**Modified non-membership values**		
0.716439	0.739569	0.739569	0.283460	0.260330	0.260330
0.728127	0.739569	0.739569	0.271772	0.260330	0.260330
0.733879	0.739569	0.739569	0.266020	0.260330	0.260330
0.739569	0.739569	0.733879	0.260330	0.260330	0.266020
0.728127	0.733879	0.733879	0.271772	0.266020	0.266020
0.716439	0.728127	0.728127	0.283460	0.271772	0.271772
0.704505	0.722314	0.722314	0.295394	0.277585	0.277585
0.704505	0.710503	0.716439	0.295394	0.289396	0.283460
0.704505	0.716439	0.722314	0.295394	0.283460	0.277585
0.710503	0.716439	0.716439	0.289396	0.283460	0.283460

Step(v): Intuitionistic defuzzification is done as shown below.

Newly generated grey level pixel values of class type double

0.582287	0.602146	0.602146
0.592214	0.602146	0.602146
0.597180	0.602146	0.602146
0.602146	0.602146	0.597180
0.592214	0.597180	0.597180
0.582287	0.592214	0.592214
0.572377	0.587249	0.587249
0.572377	0.577329	0.582287
0.572377	0.582287	0.587249
0.577329	0.582287	0.582287

New grey level values generated in step (v) gives the restored image. Hence, it is inferred that noisy white regions are suppressed by the proposed method and filtered image is obtained.



(a) (b) (c)

Figure 3.2: Final image obtained after IF median filtering. Figure (a) Input image, (b) Salt and pepper noisy image, (c) Restored image (IF median filtered)

The original lena image and salt and pepper noisy image are exhibited in Figure 3.2(a), Figure 3.2(b) and corresponding restored image (IF median filtered) is shown in Figure 3.2(c).

Example 3.5.2. In order to compare the performance of traditional, fuzzy and intuitionistic fuzzy filters, a standard cameraman image of size 256 $\times$ 256 with salt and pepper noise is taken for analysis as shown in Figure 3.3.

Figure (a)

Figure (b)

Figure 3.3: (a) Input image, (b) Salt and pepper noisy image.

The results of denoising using traditional, fuzzy and intuitionistic fuzzy filters are displayed in Figure 3.5 (p.59) in which first column refers to traditional filtered image, second and third column refers to fuzzy and intuitionistic fuzzy filtered images respectively. In order to remove impulse noise, several intuitionistic fuzzy filtering techniques are employed. Four different filtering techniques namely intuitionistic fuzzy (IF) mean, IF median, IF max and IF min filters are applied and their filtering performance on impulse noise is presented.

It is inferred that from Figure 3.5, traditional filters often tend to blur sharp edges, and affect the edge details. They remove smaller percent of noise and performance is slow in presence of high noise. Fuzzy based filtering approach perform better noise removal and have great deal with low level noise to high level noise corrupted in the images. In addition, intuitionistic fuzzy filters perform better than fuzzy by giving stability in accuracy.

3.6 Performance Analysis

IF filtering algorithm is employed to reduce noise in images and to enhance the image quality. Experimental results indicate that the proposed method performs significantly better in preserving image details and also preserving image edge information and achieves better results when applied to images corrupted by impulse noise (salt and pepper).

Performance of the filters are tested at different level of noise densities on the basis of PSNR values [50]. PSNR is usually expressed in terms of the logarithmic decibel (dB) scale.

Restored image performance is quantified using PSNR as defined below:

$$PSNR = 10log_{10} \frac{255^2}{\frac{1}{mn}\sum\limits_{i,j}(r_{i,j} - x_{i,j})^2} \tag{3.1}$$

where r - original image, x - restored image and mn - size of the image.

Comparison on the performance of proposed method to existing methods for median filter is shown in Table 3.1.

Type of filter	**Salt and pepper noise densities**			
	20 %	30 %	50 %	80 %
Adaptive median	35.57	30.72	27.37	22.39
Standard median	41.59	33.75	26.43	14.57
Fuzzy median	53.28	37.38	33.65	27.71
IF median	68.96	59.58	56.98	52.23

Table 3.1: Performance comparison of proposed method to existing methods for median filter

The noise density levels vary from 20% to 80% and the performance are quantitatively measured by PSNR. For 20% noise level, PSNR value for adaptive median filter is 35.57, for standard median and fuzzy median filter, PSNR values are 41.59 and 53.28 respectively. Here, IF median value is 68.96. Similarly, for other noise density levels, the PSNR value of IF median is higher than other filters. The higher values of PSNR infer that proposed median filter is in acceptable ratio. Hence, from Table 3.1, it is obvious that IF median filter performs better.

The graphical representation of PSNR Vs Noise densities (in %) is shown in Figure 3.4. Comparison on the performance of traditional median, fuzzy median and intuitionistic fuzzy median are plotted in the graph. The noise levels vary from 20% to 80% and the corresponding PSNR values vary from 14 to 69 dB. From Figure 3.4, it is inferred that for 20% noise level, PSNR value for standard median is 41.59 and for fuzzy median and IF median filter, it is 53.28 and 68.96 respectively.

It is observed that, increase in noise level results more stability in accuracy, comparing with other existing filtering techniques. It is evident from the graph that, the proposed IF median filter stands top in removing the noise and preserves

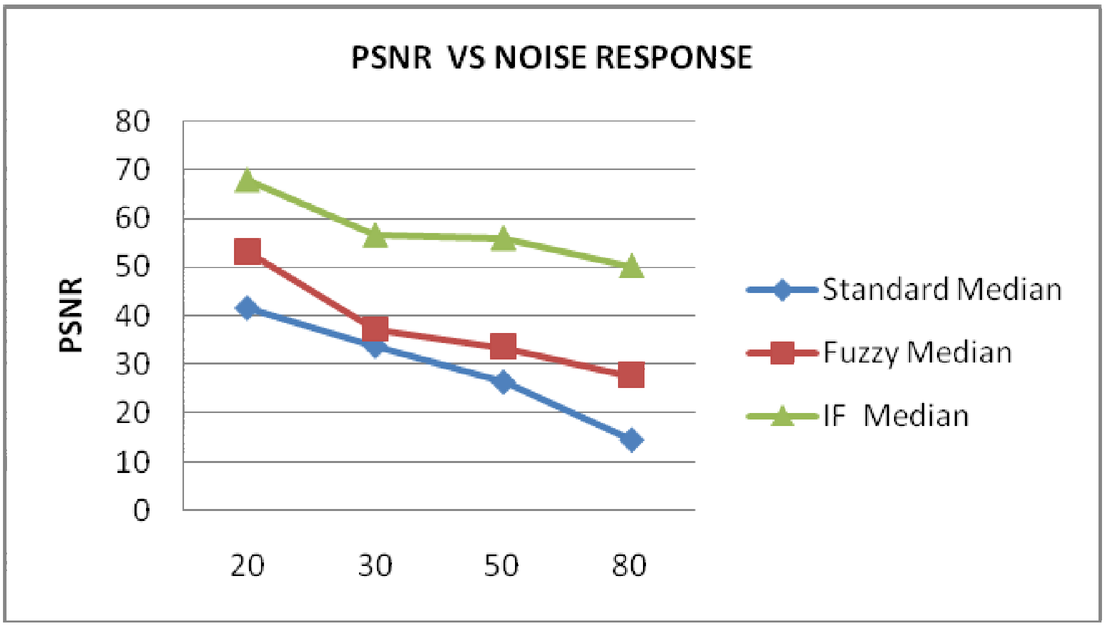


Figure 3.4: Graphical representation of PSNR Vs Noise densities (in %)

the image details.

Figure 3.5: Comparison of filters (a) traditional, (b) fuzzy and (c) intuitionistic fuzzy filters.

Chapter 4

Intuitionistic fuzzy statistical tools

4.1 Introduction

In this chapter, an attempt has been made to define intuitionistic fuzzy random variable (IFRV) and to study some of its properties. Also, characteristics of intuitionistic fuzzy random variable like expectation, variance and moments are described. Intuitionistic fuzzy number defined as a generalization of Wu's fuzzy number, can be viewed as an alternative approach to model reality with uncertainty for solving problems in IF system and can be applied to many practical problems arising on economical, social survey, pollution control.

Computation of information based on IF statistics is more futuristic and are widely used in decision making. Hence, the authors are motivated to design IF statistical tools and these new techniques can extract people's thought in a more precise way. In addition, IF statistical tools are used to design intuitionistic fuzzy filtering algorithm. The validity of the proposed algorithm, is verified with suitable examples.

4.2 Preliminaries

In this section, some basic definitions that are necessary to the study are given.

Definition 4.2.1.

Let U be the universal set and let $A = \{A_1, A_2, \cdots, A_n\}$ be the subset of discussion factors in U, then for any $x \in U$, its degrees of membership and non-membership corresponding to $\{A_1, A_2, \cdots, A_n\}$ are respectively $\{\mu_1(x), \cdots, \mu_n(x)\}$ and $\{\nu_1(x), \nu_2(x), \cdots, \nu_n(x)\}$ where $\mu : U \to [0,1]$ and $\nu : U \to [0,1]$ are real, then the *intuitionistic fuzzy number* is an object of the form $\{\langle x, \mu_U(x), \nu_U(x)\rangle : x \in U\}$ and can be written as

$$\mu_U(x) = \tfrac{\mu_1(x)}{A_1} + \tfrac{\mu_2(x)}{A_2} + \cdots + \tfrac{\mu_n(x)}{A_n},$$
$$\nu_U(x) = \tfrac{\nu_1(x)}{A_1} + \tfrac{\nu_2(x)}{A_2} +, \cdots + \tfrac{\nu_n(x)}{A_n}$$

where "+" stands for "or" and " $\div$ " stands for the membership $\mu_i(x)$ on A_i. It can also be written as $A_i(\mu, \nu) = \frac{\langle \mu_1(x), \nu_1(x)\rangle}{A_1} + \frac{\langle \mu_2(x), \nu_2(x)\rangle}{A_2} + \cdots + \frac{\langle \mu_n(x), \nu_n(x)\rangle}{A_n}$

Example 4.2.1. A survey about favourite subjects using IFN:
Consider an intuitionistic fuzzy set of *like* subjects S_1, S_2, S_3, S_4 and S_5 of a person as shown in Table 4.1. When the degree is given as 1 or 0 , that is *like* or *dislike*, a standard *Yes* or *No* are in complementary relationship as in binary logic.

Let A_1 represents *like subjects*, A_2 represents *dislike* subjects. $A_1(\mu, \nu)$ denotes IF response of *like subjects*. $A_2(\mu, \nu)$ denotes IF response of *dislike subjects*. The intuitionistic fuzzy number of these two statements can be represented as

Like Subjects	IF Perception		Binary Perception	
	$A_1(\mu,\nu)$	$A_2(\mu,\nu)$	$A_1 = like$	$A_2 = dislike$
Mathematical Science (S_1)	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.1\rangle$		Yes
Physical Science (S_2)	$\langle 0.5, 0.2\rangle$	$\langle 0.1, 0.4\rangle$		No
Social Science (S_3)	$\langle 0.6, 0.3\rangle$	$\langle 0.2, 0.6\rangle$	Yes	
Biological Science (S_4)	$\langle 0.8, 0.1\rangle$	$\langle 0.2, 0.4\rangle$	Yes	
Life Science (S_5)	$\langle 0.2, 0.6\rangle$	$\langle 0.6, 0.3\rangle$	No	

Table 4.1: Comparison of IF perception and Binary perception on favourite subjects

$$A_1(\mu,\nu) = \frac{\langle 0.1,0.8\rangle}{S_1} + \frac{\langle 0.5,0.2\rangle}{S_2} + \frac{\langle 0.6,0.3\rangle}{S_3} + \frac{\langle 0.8,0.1\rangle}{S_4} + \frac{\langle 0.2,0.6\rangle}{S_5}$$
$$A_2(\mu,\nu) = \frac{\langle 0.8,0.1\rangle}{S_1} + \frac{\langle 0.1,0.4\rangle}{S_2} + \frac{\langle 0.2,0.6\rangle}{S_3} + \frac{\langle 0.2,0.4\rangle}{S_4} + \frac{\langle 0.6,0.3\rangle}{S_5}$$

If someone uses membership and non-membership functions to express their degree of feeling based on human perception, the result will be very closer to the human thought. IFN gives a better representation of favoritism of subjects of a person by giving the degree of belongingness or non belongingness instead of Yes or No answer.

Definition 4.2.2. [9]

Let $A(\mu,\nu)$ be an IFN. Then $S(A) = \mu - \nu$ is called as the *score* of A where $S(A) \in [-1, 1]$.

Definition 4.2.3. [9]

Let $A(\mu, \nu)$ be an IFN. Then $L(A) = \mu + \nu$ be the *accuracy* of A where $L(A) \in [0, 1]$.

Note

Let A and B be any two IFNs. According to their scores and accuracies, the ranking order [57] of A and B is stipulated as follows:

(i) If $S(A) > S(B)$, then A is greater than B, denoted by $A \succ B$.

(ii) If $S(A) < S(B)$, then A is smaller than B, denoted by $A \prec B$.

(iii) If $S(A) = S(B)$, then there arises three situations:

(a) If $L(A) = L(B)$, then A is equal to B, denoted by $A = B$.

(b) If $L(A) > L(B)$, then A is greater than B, denoted by $A \succ B$.

(c) If $L(A) < L(B)$, then A is smaller than B, denoted by $A \prec B$.

Note

Let $A = ([a, b], [c, d])$, denote an interval valued intuitionistic fuzzy number (IV-IFN).

Definition 4.2.4. [44]

Let $A_i = ([a_i, b_i], [c_i, d_i])$ be an IVIFN, accuracy function L of A_i is defined as $L(A_i) = \frac{a_i + b_i - d_i(1-b_i) - c_i(1-a_i)}{2}$.

Definition 4.2.5. [44]

Let $A_i = ([a_i, b_i], [c_i, d_i])$ be an IVIFN. The following are the steps involved in ranking interval-valued intuitionistic fuzzy numbers.

Step 1: Find the score function of A_i using $L(A_i) = \frac{a_i + b_i - d_i(1-b_i) - c_i(1-a_i)}{2}$.

Step 2: Calculate $B_{A_i(x_i,x_j)} = \{a_i \in A_i/x_i > x_j\}$,

$C_{A_i(x_i,x_j)} = \{a_i \in A_i/x_i = x_j\}$ and fuzzy dominance relation using the formula $R_{A_i(x_i,x_j)} = \frac{\left|B_{A_i(x_i,x_j)}\right|+\left|C_{A_i(x_i,x_j)}\right|}{2|A_i|}$.

Step 3: Calculate the entire dominance degree using $R_{A_i(x_i)} = \frac{1}{|E|}\sum_{i=1}^{|E|} R_{A_i(x_i,x_j)}$.

Step 4: Now, the IVIFNs are ranked based on the value of entire dominance degree.

4.3 Intuitionistic fuzzy random variable

Definition 4.3.1.

Let U be the universal set and let $L = \{L_1, L_2, \cdots, L_k\}$ be the set of k linguistic variables. Then an *intuitionistic fuzzy random variable* X is characterized by the map $X : U \to F$ such that $x_i \in U$ is associated with an ordered pair in $[0,1]\times[0,1]$ of the form $F = \langle \mu_{ij}, \nu_{ij}\rangle$, where the function $\mu_{ij} : U \to [0,1]$ and $\nu_{ij} : U \to [0,1]$ define the degrees of membership and non-membership of x_i with respect to L_j, such that $0 \leq \mu_{ij} + \nu_{ij} \leq 1$.

Example 4.3.1. A farmer wants to adapt a new farming style for cultivating a crop from traditional techniques. He invites 5 experts for evaluation. After they tested the crop, they are asked to give a IF grading as Very Unsatisfactory $= L_1$, Unsatisfactory$= L_2$, No difference$= L_3$, Satisfactory$= L_4$, Very Satisfactory$= L_5$. Table 4.2 shows the evaluation of the 5 experts.

Expert	$\boldsymbol{L_1}$	$\boldsymbol{L_2}$	$\boldsymbol{L_3}$	$\boldsymbol{L_4}$	$\boldsymbol{L_5}$
A	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.1\rangle$	–	–	$\langle 0.2, 0.6\rangle$
B	$\langle 0.1, 0.6\rangle$	–	$\langle 0.4, 0.1)$	$\langle 0.7, 0.2\rangle$	–
C	–	$\langle 0.2, 0.6\rangle$	–	$\langle 0.4, 0.3)$	$\langle 0.7, 0.2\rangle$
D	$\langle 0.8, 0.1\rangle$	$\langle 0.5, 0.2)$	–	$\langle 0.2, 0.6\rangle$	–
E	$\langle 0.1, 0.6\rangle$	–	–	$\langle 0.7, 0.1\rangle$	$\langle 0.8, 0.2\rangle$

Table 4.2: An example of IFRV

Though, the responses are available, still there remains an uncertainty about precise meaning of the response. IFRV extends the case to model not only uncertainty but also to model the hesitation which is naturally present in the uncertainty.

4.4 Properties of IFRV

1. Let U be the universal set. Let X_1 and X_2 be the set of two IFRVs over X, where $X_1 = \{\langle x_i, L_k, \mu_{ik}, \nu_{ik}\rangle : x_i \in X\}$ and $X_2 = \left\{\left\langle x_j, L_k, \mu_{jk}, \nu_{jk}\right\rangle : x_j \in X\right\}$, $i = 1, 2, ..., m$ and $k = 1, 2, ..., n$ then

   (a) $X_1 + X_2$ is an object of the form

   $X_1 + X_2 = \{\langle x_i + x_j, L_k, \mu_{ij}, \nu_{ij}\rangle : x_i, x_j \in X\}$, where $\mu_{ij} = \mu_{ik}(x_i) + \mu_{jk}(x_j) - \mu_{ik}(x_i).\mu_{jk}(x_j), \nu_{ij} = \nu_{ik}(x_i).\nu_{jk}(x_j)$ and $0 \leq \mu_{ij} + \nu_{ij} \leq 1$ is also an IFRV.

(b) $X_1.X_2 = \{\langle x_i.x_j, L_k, \mu_{ij}, \nu_{ij}\rangle : x_i, x_j \in X\}$, where $\mu_{ij} = \mu_{ik}(x_i)+\mu_{jk}(x_j)$ and $\nu_{ij} = \nu_{ik}(x_i) + \nu_{jk}(x_j) - \nu_{ik}(x_i).\nu_{jk}(x_j)$ is also an IFRV.

2. If X is an IFRV and n is a scalar, then $nX = \{\langle x, 1 - (1 - \mu_{ik}(x))^n, \nu_{ik}(x)^n\rangle\}$ is also an IFRV.

3. If X_1 and X_2 are IFRVs, then $max(X_1, X_2)$ and $min(X_1, X_2)$ are also IFRVs, where

$$max(X_1, X_2) = \Big\{\Big\langle x, max(\mu_{ik}(x_i), \mu_{jk}(x_j)), min(\nu_{ik}(x_i), \nu_{jk}(x_j))\Big\rangle\Big\} \text{ and}$$
$$min(X_1, X_2) = \Big\{\Big\langle x, min(\mu_{ik}(x_i), \mu_{jk}(x_j)), max(\nu_{ik}(x_i), \nu_{jk}(x_j))\Big\rangle\Big\}.$$

Definition 4.4.1. [34]

Let X be an intuitionistic fuzzy random variable. Then the function F defined for all x by $P(x) = \sum\limits_{x_i<x} p_i$ for $i = 1, 2, ..., n$ is called the distribution function of the IFRV.

Any set of pairs $\{(\frac{1}{2}(\mu_A(x_i) + 1 - \nu_A(x_i)).x_i, p_i), i = 1, 2, ..., n\}$ will be the probability distribution of an IFRV.

Definition 4.4.2.

Let X be an IFRV over U. Then its *expected value* is defined as

$$E(X) = \sum_{i=1}^{n} \tfrac{1}{2}(\mu_A(x_i) + 1 - \nu_A(x_i)).x_i.p_i$$

Remark

$$E(X^2) = \sum_{i=1}^{n} \tfrac{1}{2}(\mu_A(x_i) + 1 - \nu_A(x_i)).x_i^2.p_i$$

Definition 4.4.3.

Let X be an IFRV over U. Then its *variance* is defined, in the usual way, as $V(X) = E(X^2) - (E(X))^2$.

Example 4.4.1. Let X be an IFRV with the following probability distribution. Find $E(X)$ and $E(X^2)$. Evaluate $E(2X+1)^2$ and $V(2X+1)^2$.

	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$
$\boldsymbol{X}$	$\langle -3, 0.6, 0.3\rangle$	$\langle 6, 0.4, 0.3\rangle$	$\langle 9, 0.7, 0.2\rangle$
$\boldsymbol{P(X)}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Here, $E(X) = 3.8675$, $E(X^2) = 31.125$, $V(X) = 16.175$

$E(2X+1)^2 = 4E(X^2) + 4E(X) + 1$

Therefore, $E(2X+1)^2 = 140.97$

$V(2X+1)^2 = a^2V(X) = 4V(X) = 64.7$

Note

1. If X is an IFRV, A is any IF event and a and b are constants, then $E_A(aX + b) = aE_A(X) + b$

Proof

$E_A(aX+b) = \sum \frac{\mu_A(x)+1-\nu_A(x)}{2}.(ax+b).p(x)$

$= \sum \frac{\mu_A(x)+1-\nu_A(x)}{2}.ax.p(x) + \frac{\mu_A(x)+1-\nu_A(x)}{2}.b.p(x)$

$E_A(aX+b) = aE_A(X) + b$

2. If $b = 0$ then $E_A(aX) = aE_A(X)$

3. If $a = 1, b = -\bar{X} = -E_A(X)$ then $E_A(X - \bar{X}) = 0$

4. $|E_A(X)| \leq E_A|X|$, provided expectation exists.

5. $E_A(\frac{1}{X}) \geq \frac{1}{E_A(X)}$

6. $E_A(X^2) \geq E_A(X)^2$

7. $E_A(X^{\frac{1}{2}}) \leq E(X)^{\frac{1}{2}}$

8. $E(logX) \leq logE(X)$

Properties of Variance

1. If X is an IFRV, A is any IF event then $V_A(aX+b) = a^2V_A(X)$, where a and b are constants.

2. When $b = 0$, $V_A(aX) = a^2V_A(X)$.

Definition 4.4.4.

The r^{th} moment of an IFRV X, denoted by $E(X^r)$, is defined as $\mu_r^{'} = E(X^r)$ that is, $E(X^r) = \sum_{i=1}^{n} \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}.x_i^r.p_i$
when $r = 0$, $\mu_0^{'} = E(1) = 1$

when $r = 1$, $\mu_1^{'} = E(X) = \mu$, the mean of the distribution.

Definition 4.4.5.

If X is an IFRV, then $\mu_r = E[(X-c)^r]$ is called the r^{th} *central moment about the mean* c where $E[(X-c)^r] = \sum_{i=1}^{n} \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}.(x_i - c)^r.p_i$

Definition 4.4.6.

Let X be a discrete IFRV with probability function p_i, then the function $M_x(t) = E(e^{tx})$ is called the *moment generating function* of X, defined by

$$M_x(t) = \sum_{i=1}^{n} \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}.e^{tx_i}.p_i$$

Properties of MGF

1. Let X be an IFRV and c be any constant, then $M_{cx}(t) = M_x(ct)$

   $M_{cx}(t) = Ee^{cxt} = Ee^{xct} = M_x(ct)$

2. $M_{c+x}(t) = e^{ct}.M_x(t)$

3. If $Y = aX + b$, then $M_Y(t) = e^{bt}.M_x(at)$

Example 4.4.2. Let X be an IFRV. The probability distribution of X is given as follows. Find the MGF of X and also find the mean and variance.

Values of X	$\langle 0, 0.3, 0.6\rangle$	$\langle 1, 0.7, 0.2\rangle$	$\langle 2, 0.5, 0.2\rangle$
$\boldsymbol{P(x)}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\boldsymbol{e^{tx}}$	1	e^t	e^{2t}

$M_x(t) = E(e^{tx}) = \sum e^{tx_i}.p_i \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}$

$M_x(t) = \frac{1}{4}(0.35 + 2.e^t \times 0.75 + 0.65.e^{2t})$

$M_x^{'}(t) = \frac{1}{2}(0.75e^t + 0.65e^{2t})$

$\mu_1^{'} = M_x^{'}(0) = \frac{1}{2}(0.75 + 0.65) = 0.7$

$\mu_2^{'} = M_x^{''}(t) = \frac{1}{2}(0.75e^t + 2e^{2t} \times 0.65)$

$M_x^{''}(0) = 1.025$

$Mean = \mu = \mu_1^{'} = 0.7$

$Variance = \mu_2^{'} - \mu_1^{'2} = 1.025 - 0.7^2 = 0.535$

Therefore, the mean of X values for the given data $= 0.7$ and the variance is 0.535.

Definition 4.4.7.

If X and Y are two IFRVs, A is any IF event, then *Covariance* between them is defined as $Cov(X, Y) = E(XY) - E(X)E(Y)$

Note

If X and Y are two independent IFRVs, then $E(XY) = E(X)E(Y)$.

Cauchy-Schwartz inequality for IFRV

Theorem

If X and Y are real - valued IFRVs, then $[E(XY)]^2 \leq E(X^2)E(Y^2)$

Proof

Consider $Z(t)$, a real valued function of a real variable t, defined by $Z(t) = E(X+tY)^2$.

$$\begin{aligned} Z(t) &= E(X+tY)^2 \\ &= E[X^2 + 2tXY + t^2Y^2] \end{aligned}$$

$Z(t) = E(X^2) + 2tE(XY) + t^2(E(Y^2)) \geq 0$ for all t, which is a quadratic equation in t of the form $\psi(t) = At^2 + Bt + c \geq 0$ for all t.

This implies that graph of the function $\psi(t)$, for positive X, Y and t, either touches the t-axis at only one point or not at all. Therefore, if $B^2 - 4AC > 0$, then the function $\psi(t)$ has two distinct real roots which means, the graph of $\psi(t)$ meets t axis at two different points which is a contradiction.

Hence, $B^2 - 4AC \leq 0$.

$\Rightarrow 4E(XY)^2 - 4E(X^2)E(Y^2) \leq 0$

$\Rightarrow E(XY)^2 \leq E(X^2)E(Y^2)$

4.5 Intuitionistic fuzzy statistical tools

Let U be the universal set, $L = \{L_1, L_2, \cdots, L_k\}$ be a set of k linguistic variables on U and let X be an intuitionistic fuzzy random variable.

Throughout this section, the notation $\frac{\langle \ldots, \ldots \rangle}{L_j}$ is used, it denotes the degrees of membership and non-membership of the linguistic variable L_j.

Definition 4.5.1.

The *intuitionistic fuzzy mean* of X, denoted by IF_M, is defined as

$$IF_M = \frac{\left\langle \frac{\sum_{i=1}^{n} \mu_{i1}}{n}, \frac{\sum_{i=1}^{n} \nu_{i1}}{n} \right\rangle}{L_1} + \frac{\left\langle \frac{\sum_{i=1}^{n} \mu_{i2}}{n}, \frac{\sum_{i=1}^{n} \nu_{i2}}{n} \right\rangle}{L_2} + \cdots + \frac{\left\langle \frac{\sum_{i=1}^{n} \mu_{ik}}{n}, \frac{\sum_{i=1}^{n} \nu_{ik}}{n} \right\rangle}{L_k}$$

Note

Since $0 \leq \mu_{ij} \leq 1$ and $0 \leq \nu_{ij} \leq 1$ for all i and j, $0 \leq \frac{\sum_{i=1}^{n} \mu_{ij}}{n} \leq 1$ and $0 \leq \frac{\sum_{i=1}^{n} \nu_{ij}}{n} \leq 1$ and hence $\frac{\sum_{i=1}^{n} \mu_{ij}}{n} + \frac{\sum_{i=1}^{n} \nu_{ij}}{n} \in [0,1]$. Hence, IF_M is also an IFS.

Example 4.5.1. A manufacturing company is launching new product for the forthcoming year. The company asked seven experts to give their grading based on the quality of the product. The quality of the product are classified as $E =$ Excellent, $G =$ Good, $F =$ Fair, $B =$ Bad, $W =$Worst. Find the expected value of the product, based on quality criteria.

Let $L = \{E, G, F, B, W\}$ be the set of the linguistic variables. Then IF expected value is given by

$$\begin{aligned} IF_M &= \frac{\left\langle \frac{2.2}{5}, \frac{1.0}{5} \right\rangle}{E} + \frac{\left\langle \frac{1.5}{5}, \frac{1.4}{5} \right\rangle}{G} + \frac{\left\langle \frac{1.5}{5}, \frac{1.3}{5} \right\rangle}{F} + \frac{\left\langle \frac{2.1}{5}, \frac{2.4}{5} \right\rangle}{B} + \frac{\left\langle \frac{2.5}{5}, \frac{1.3}{5} \right\rangle}{W} \\ &= \frac{\langle 0.44, 0.14 \rangle}{E} + \frac{\langle 0.21, 0.2 \rangle}{G} + \frac{\langle 0.21, 0.18 \rangle}{F} + \frac{\langle 0.3, 0.34 \rangle}{B} + \frac{\langle 0.35, 0.18 \rangle}{W} \end{aligned}$$

Expert analysis	E	G	F	B	W
$\boldsymbol{E_1}$	$\langle 0.6, 0.2\rangle$	$\langle 0.4, 0.1\rangle$	$\langle 0.3, 0.3\rangle$	$-$	$-$
$\boldsymbol{E_2}$	$-$	$-$	$\langle 0.4, 0.1\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.8, 0.1\rangle$
$\boldsymbol{E_3}$	$-$	$\langle 0, 0.8\rangle$	$\langle 0.1, 0.2\rangle$	$\langle 0.3, 0.2\rangle$	$\langle 0.7, 0.2\rangle$
$\boldsymbol{E_4}$	$\langle 0.1, 0.6\rangle$	$-$	$-$	$\langle 0.5, 0.2\rangle$	$\langle 0.3, 0.1\rangle$
$\boldsymbol{E_5}$	$\langle 0.8, 0.1\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.4, 0.1\rangle$	$\langle 0.1, 0.7\rangle$	$-$
$\boldsymbol{E_6}$	$-$	$-$	$\langle 0.3, 0.6\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.6, 0.1\rangle$
$\boldsymbol{E_7}$	$\langle 0.7, 0.1\rangle$	$\langle 0.5, 0.2\rangle$	$-$	$\langle 0.2, 0.7\rangle$	$\langle 0.1, 0.8\rangle$
$\boldsymbol{\langle \sum_{i=1}^{5} \mu_{i1}, \sum_{i=1}^{5} \nu_{i1}\rangle}$	$\langle 2.2, 1.0\rangle$	$\langle 1.5, 1.4\rangle$	$\langle 1.5, 1.3\rangle$	$\langle 2.1, 2.4\rangle$	$\langle 2.5, 1.3\rangle$
$\boldsymbol{IF_M}$	$\langle 0.44, 0.14\rangle$	$\langle 0.21, 0.2\rangle$	$\langle 0.21, 0.18\rangle$	$\langle 0.3, 0.34\rangle$	$\langle 0.35, 0.18\rangle$
Score function	0.3	0.01	0.03	-0.04	0.17

Table 4.3: Expert analysis on quality

Based on the values of the score function of IF_M, it is expected that quality of the product is *Excellent* and the company is supposed to continue the production retaining the same quality.

Definition 4.5.2.

Let X be an intuitionistic fuzzy random variable. Then

(i) *intuitionistic fuzzy geometric mean* of X, denoted by $IF_{G.M}$, is defined as

$$IF_{G.M} = \frac{\left\langle (\prod_{i=1}^{n} \mu_{i1})^{\frac{1}{n}}, 1-(\prod_{i=1}^{n} (1-\nu_{i1}))^{\frac{1}{n}} \right\rangle}{L_1} + \frac{\left\langle (\prod_{i=1}^{n} \mu_{i2})^{\frac{1}{n}}, 1-(\prod_{i=1}^{n} (1-\nu_{i2}))^{\frac{1}{n}} \right\rangle}{L_2} + \cdots + \frac{\left\langle (\prod_{i=1}^{n} \mu_{ik})^{\frac{1}{n}}, 1-(\prod_{i=1}^{n} (1-\nu_{ik}))^{\frac{1}{n}} \right\rangle}{L_k}$$

(ii) *intuitionistic fuzzy harmonic mean* of X, denoted by $IF_{H.M}$, is defined as

$$IF_{H.M} = \frac{\left\langle \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{\mu_{i1}}}, 1 - \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{1-\nu_{i1}}} \right\rangle}{L_1} + \frac{\left\langle \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{\mu_{i2}}}, 1 - \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{1-\nu_{i2}}} \right\rangle}{L_2} + \cdots + \frac{\left\langle \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{\mu_{ik}}}, 1 - \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{1-\nu_{ik}}} \right\rangle}{L_k}$$

Example 4.5.2. A production firm wants to launch a new cosmetic item. The company director asks five experts to grade after introducing the product with the classification of the profit as HS = Highly Satisfied, S = Satisfied, M = Moderate, BM = Below Moderate, DS = Dis Satisfied. The opinion of the experts is given as IF grading. Give your suggestions to the company to launch the new product based on IF geometric mean and IF harmonic mean.

Here $L = \{HS, S, M, BM, DS\}$ is the set of linguistic variables.

Expert analysis	**HS**	**S**	**M**	**BM**	**DS**
$\boldsymbol{E_1}$	$\langle 0.8, 0.1 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.2, 0.4 \rangle$	$\langle 0.2, 0.7 \rangle$
$\boldsymbol{E_2}$	$\langle 0.6, 0.2 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.4, 0.3 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.1, 0.8 \rangle$
$\boldsymbol{E_3}$	$\langle 0.1, 0.8 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.4, 0.2 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.8, 0.1 \rangle$
$\boldsymbol{E_4}$	$\langle 0.2, 0.7 \rangle$	$\langle 0.1, 0.6 \rangle$	$\langle 0.3, 0.2 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.6, 0.3 \rangle$
$\boldsymbol{E_5}$	$\langle 0.8, 0.1 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.3, 0.3 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.1, 0.7 \rangle$
$\boldsymbol{IF_{G.M}}$	$\langle 0.37, 0.5 \rangle$	$\langle 0.3, 0.4 \rangle$	$\langle 0.37, 0.26 \rangle$	$\langle 0.28, 0.45 \rangle$	$\langle 0.2, 0.6 \rangle$
Score function	-0.13	-0.1	0.11	-0.17	-0.4
$\boldsymbol{IF_{H.M}}$	$\langle 0.26, 0.5 \rangle$	$\langle 0.24, 0.4 \rangle$	$\langle 0.36, 0.26 \rangle$	$\langle 0.27, 0.5 \rangle$	$\langle 0.2, 0.6 \rangle$
Score function	-0.31	-0.2	0.1	-0.23	-0.4

Table 4.4: Opinion of experts on launching new products

Then IF geometric Mean is

$$IF_{G.M} = \frac{\left\langle (0.007)^{\frac{1}{5}}, 1-(0.38)^{\frac{1}{5}} \right\rangle}{HS} + \frac{\left\langle (0.003)^{\frac{1}{5}}, 1-(0.07)^{\frac{1}{5}} \right\rangle}{S} + \frac{\left\langle (0.007)^{\frac{1}{5}}, 1-(0.021)^{\frac{1}{5}} \right\rangle}{M}$$

$$+ \frac{\left\langle (0.002)^{\frac{1}{5}}, 1-(0.04)^{\frac{1}{5}} \right\rangle}{BM} + \frac{\left\langle (0.0009)^{\frac{1}{5}}, 1-(0.011)^{\frac{1}{5}} \right\rangle}{DS}$$

$$= \frac{\langle 0.37, 0.5 \rangle}{HS} + \frac{\langle 0.3, 0.4 \rangle}{S} + \frac{\langle 0.37, 0.26 \rangle}{M} + \frac{\langle 0.28, 0.45 \rangle}{BM} + \frac{\langle 0.24, 0.45 \rangle}{DS}$$

and IF harmonic Mean is

$$IF_{H.M} = \frac{\left\langle \frac{5}{19.1}, 1-\frac{5}{11.8} \right\rangle}{HS} + \frac{\left\langle \frac{5}{20.66}, 1-\frac{5}{8.92} \right\rangle}{S} + \frac{\left\langle \frac{5}{13.66}, 1-\frac{5}{6.78} \right\rangle}{M} + \frac{\left\langle \frac{5}{18.42}, 1-\frac{5}{10} \right\rangle}{BM}$$

$$+ \frac{\left\langle \frac{5}{27.9}, 1-\frac{5}{14.2} \right\rangle}{DS}$$

$$= \frac{\langle 0.26, 0.57 \rangle}{HS} + \frac{\langle 0.24, 0.44 \rangle}{S} + \frac{\langle 0.36, 0.26 \rangle}{M} + \frac{\langle 0.27, 0.5 \rangle}{BM} + \frac{\langle 0.17, 0.64 \rangle}{DS}$$

Based on the values of the score function S_j of $IF_{G.M}$ and $IF_{H.M}$, it is inferred that after introducing the new product, the company will get a *moderate* profit. Hence, it is suggested to launch the new product.

Definition 4.5.3.

Let U be the universal set. Let X be an IFRV on U and let $L = \{L_1, L_2, \cdots, L_k\}$ be a set of k linguistic variables. Denote $I_j = \sum_{i=1}^{n} \mu_{ij}$ and $J_j = \sum_{i=1}^{n} \nu_{ij}$. Assume that $\langle L_j, NI_j, NJ_j \rangle$ denote the normalised sum of membership and non-membership of I_j and J_j with respect to L_j such that $NI_j = \frac{I_j}{supI_j}$ and $NJ_j = \frac{J_j}{supJ_j}$. Let S_j be the score function. Then

(i) *intuitionistic fuzzy median* of X is defined as the median of S_j.

That is, $IF_{Med} = L_j$ corresponding to median of S_j.

(ii) *intuitionistic fuzzy mode* of X is defined as the maximum of S_j.

That is, $IF_{Mo} = L_j$ corresponding to maximum of S_j.

Example 4.5.3. In a newly started Diet centre, a study was made to analyse the diets. Five experts are asked to evaluate the nature of diet of a person. The table below shows the diet analysis by experts. L_1 = Very healthy, L_2 = Healthy, L_3 = Normal, L_4 = Weak, L_5 = Poor be the set of linguistic variables. Find the IF median and IF mode for the diet of a person.

Diet analysis	L_1	L_2	L_3	L_4	L_5
D_1	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.3\rangle$	$-$	$\langle 0.1, 0.7\rangle$	$-$
D_2	$-$	$-$	$\langle 0.4, 0.2\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.8, 0.1\rangle$
D_3	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.3, 0.1\rangle$	$\langle 0.2, 0.7\rangle$	$-$
D_4	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.1\rangle$	$\langle 0.3, 0.2\rangle$	$-$	$\langle 0.2, 0.7\rangle$
D_5	$-$	$\langle 0.1, 0.6\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.8, 0.1\rangle$
$\langle I_j, J_j\rangle$	$\langle 2.1, .5\rangle$	$\langle 1.9, 1.1\rangle$	$\langle 1.5, 0.8\rangle$	$\langle 1.5, 1.6\rangle$	$\langle 1.8, 0.9\rangle$
Normalized Sum	$\langle 1.0, 0.3\rangle$	$\langle 0.9, 0.6\rangle$	$\langle 0.7, 0.5\rangle$	$\langle 0.7, 1.0\rangle$	$\langle 0.9, 0.5\rangle$
Score	0.7	0.3	0.2	-0.3	0.4

Table 4.5: Opinion of five experts about the diet

In analysing the nature of diet of a person, *Very Healthy* and *Healthy* are the most analysed factor. *Healthy* is the IF Median and *Very Healthy* is the IF mode which suggest that the diet of a person is Healthy and he is advised to continue the same diet.

Definition 4.5.4.

Let U be the universal set. Let X be an IFRV on U and let $L = \{L_1, L_2, \cdots, L_k\}$ be a set of k linguistic variables. Denote $I_j = \sum\limits_{i=1}^{n} \mu_{ij}$ and $J_j = \sum\limits_{i=1}^{n} \nu_{ij}$. Assume that $\langle L_j, NI_j, NJ_j \rangle$ denote the normalised sum of membership and non-membership of I_j and J_j with respect to L_j such that $NI_j = \frac{I_j}{supI_j}$ and $NJ_j = \frac{J_j}{supJ_j}$. Let S_j be the score function and let w_j be the weights. Then *intuitionistic fuzzy weighted mean* of X, denoted by IF_W is defined as $IF_W = \langle \frac{\prod\limits_{j=1}^{n} W_j S_j}{L_j} \rangle$

Example 4.5.4. An organisation is analysing the economical condition of a country for the next period. Assume that the company operating in Europe and South Asia is analysing its general policy for next year, based on some strategy. The group of experts of the company considers the economical situation as the key factor. Depending on the situation, the expected benefits for the company will be different. The experts have considered five possible situations for the next year as S_1 = Excellent, S_2 = Good, S_3 = Fair, S_4 = Bad, S_5 = Worst. Find the weighted arithmetic mean for the analysing factor of the company.

Assume that the experts use the weight function for the calculation. Economical condition is the influencing factor for the progress of the company.

Expert analysis	S_1	S_2	S_3	S_4	S_5
D_1	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.1\rangle$	$\langle 0.3, 0.2\rangle$	$-$	$\langle 0.2, 0.7\rangle$
D_2	$\langle 0.8, 0.1\rangle$	$\langle 0.7, 0.3\rangle$	$-$	$\langle 0.1, 0.7\rangle$	$-$
D_3	$-$	$\langle 0.1, 0.6\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.8, 0.1\rangle$
D_4	$-$	$-$	$\langle 0.4, 0.2\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.8, 0.1\rangle$
D_5	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.3, 0.1\rangle$	$\langle 0.2, 0.7\rangle$	$-$
$\langle I_j, J_j\rangle$	$\langle 2.1, 0.5\rangle$	$\langle 1.9, 1.1\rangle$	$\langle 1.5, 0.8\rangle$	$\langle 1.5, 1.6\rangle$	$\langle 1.8, 0.9\rangle$
Normalized Sum	$\langle 1.0, 0.3\rangle$	$\langle 0.9, 0.6\rangle$	$\langle 0.7, 0.5\rangle$	$\langle 0.7, 1.0\rangle$	$\langle 0.9, 0.5\rangle$
Score	0.7	0.3	0.2	-0.3	0.4
Weights	9	8	7	6	7
IFW_{mean}	1.2	0.48	.28	-0.36	0.64

Table 4.6: Expert analysis about economical condition of the company

From Table 4.6, it is infered that the economical condition of the company is *Excellent.* There will be a hike in the economical condition for the next year if the company follows the existing strategies.

4.6 IF statistical tools for IVIFNs

Definition 4.6.1.

Let U be the universal set and $X = \{\langle x_i, [a_i, b_i], [c_i, d_i]\rangle : x_i \in U, a_i, b_i, c_i, d_i \in \Re\}$ where $i = 1, 2, \cdots, n$ be the set of IVIFNs, on U, then the *intuitionistic fuzzy mean* of IVIFNs denoted by $IVIF\bar{x}$ is defined as

$$IVIF\bar{x} = \left\{\left\langle x_i, \left[\frac{\sum_{i=1}^{n} a_i}{n}, \frac{\sum_{i=1}^{n} b_i}{n}\right], \left[\frac{\sum_{i=1}^{n} c_i}{n}, \frac{\sum_{i=1}^{n} d_i}{n}\right] : x_i \in X\right\rangle\right\}$$

Notations

Let U be the universal set, $L = \{L_1, L_2, \cdots, L_k\}$ be a set of k linguistic variables on U and let $X = \{\langle x_i, [a_i, b_i], [c_i, d_i]\rangle : x_i \in U, a_i, b_i, c_i, d_i \in \Re\}$ where $i = 1, 2, \cdots, n$ be the set of IVIFNs on U. Let $\rho(x_j)$ be the ranks of IVIFNs x_j.

Definition 4.6.2.

The *intuitionistic fuzzy median* of X is defined as the median of $\rho(x_j)$. That is, $IVIF_{Med} = L_j$ corresponding to median of $\rho(x_j)$.

Definition 4.6.3.

The *intuitionistic fuzzy mode* of X is defined as the maximum of $\rho(x_j)$. That is, $IVIF_{Mo} = L_j$ corresponding to maximum of $\rho(x_j)$.

Example 4.6.1. An enterprise plans to seek an adequate supplier for purchasing equipments needed for assembling the parts. Consider a problem of selection of the best supplier among five suppliers based on five attributes. Let L_1= Quality of the product, L_2= Social involvement, L_3 = Performance of delivery, L_4 = Legal issue, L_5 = Customer relationship be the five linguistic variables. Find also the $IVIF_{Med}$ and $IVIF_{Mode}$ for the best supplier.

Expert analysis	L_1	L_2	L_3	L_4	L_5
x_1	$\langle[0, 0.2], [0.2, 0.4]\rangle$	$\langle 0, [0.4, 0.6]\rangle$	$\langle[0.2, 0.4], [0.4, 0.6]\rangle$	$\langle 0.4, [0.2, 0.4]\rangle$	$\langle 0.2, [0.4, 0.8]\rangle$
x_2	$\langle 0, [0, 0.6]\rangle$	$\langle 0, 1\rangle$	$\langle[0.2, 0.4], [0.4, 0.6]\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.4, 0.2\rangle$
x_3	$\langle 0.6, 0.4\rangle$	$\langle[0.4, 0.6], [0.2, 0.4]\rangle$	$\langle 0.8, 0\rangle$	$\langle[0.2, 0.4], [0, 0.4]\rangle$	$\langle[0.4, 0.6], [0, 0.2]\rangle$
x_4	$\langle[0, 0.2], [0.2, 0.4]\rangle$	$\langle[0.2, 0.4], [0, 0.4]\rangle$	$\langle[0.2, 0.6], [0, 0.2]\rangle$	$\langle[0.2, 0.4], [0.4, 0.6]\rangle$	$\langle[0.4, 0.6], [0, 0.2]\rangle$
x_5	$\langle 0.2, 0.8\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.4, 0.4\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.4, 0.2\rangle$

Table 4.7: Evaluation of suppliers with respect to five attributes

Step1: To find $R_A(x_i, x_j)$

$\boldsymbol{R_A(x_i, x_j)}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$
$\boldsymbol{x_1}$	0.5	0.7	0.2	0.3	0.4
$\boldsymbol{x_2}$	0.3	0.5	0	0	0.5
$\boldsymbol{x_3}$	0.8	1	0.5	0.9	1
$\boldsymbol{x_4}$	0.7	1	0.1	0.5	1
$\boldsymbol{x_5}$	0.6	0.5	0	0	0.5

Step2: Calculate Rank

$\boldsymbol{x_i}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$
$\boldsymbol{R_A(x_i)}$	0.42	0.28	0.84	0.66	0.32
$\boldsymbol{\rho(x_i)}$	3	1	5	4	2

Step 3: $IF_{Med} = L_1/\rho(x_j) =$ median of $\rho(x_i)$

$IF_{Med} = 0.42$ corresponding to quality of the product.

$IF_{Mo} = L_3/\rho(x_j) =$ maximum of $\rho(x_i)$

$IF_{Mo} = 0.84$ corresponding to performance of delivery.

Hence, it is inferred

1. from the value of IF_{Median} that the best supplier will be selected based on the *Quality of the product*, they supply.

2. from IF_{Mode} that the best supplier will be selected based on the *Performance of delivery.*

4.7 IF filters in image processing

IF filters in image processing involves a set of operations using the concept of IFS theory. IF filtering algorithms are designed for noise suppression and to enhance the quality of the affected image. The performance of the proposed method is tested in MATLAB simulations for an image that has been subjected to various noise [118]. A comparative analysis illustrate the effectiveness of the algorithm. The existing filtering techniques in image processing helps to enhance the image using only the membership values. To improve the output image, the operators on IFSs are used to develop a new system. The aim of this new system is to filter the noise in the image and to enhance image quality. In this section, the proposed algorithm for IF filters is discussed:

(i) Read the noisy image and obtain the grey level matrix.

(ii) Set the parameters F_e, F_d as

$$F_e = 1,\ F_d = \frac{X_{max} - X_{min}}{(0.5)^{-1/F_e} - 1}$$

where X_{max} is the maximum level obtained in step(i).

(iii) Define the membership function

$$\mu_{mn} = \left[1 + \frac{X_{max} - X_{mn}}{F_d}\right]^{-F_e}$$

where X_{mn} is the grey level of the pixel which is to be fuzzified.

(iv) Calculate the non-membership values

$$\nu_{mn} = \frac{1}{2}\max[|1 - \mu_{mn}|, |0 - \mu_{mn}|] \qquad \text{if } 0 \leq \mu_{mn} \leq 0.5$$
$$\nu_{mn} = \frac{1}{2}\min[|1 - \mu_{mn}|, |0 - \mu_{mn}|] \qquad \text{if } 0.5 \leq \mu_{mn} \leq 1$$

such that $0 \leq \mu_{mn} + \nu_{mn} \leq 1$.

(v) Modify membership and non-membership values $\langle \mu'_{mn}, \nu'_{mn} \rangle$ using any one of the intuitionistic fuzzy filters namely IF mean, IF maximum, IF minimum, IF median [using Definitions 3.2.1, 3.2.5, 3.2.6, 3.2.2] respectively.

(vi) Calculate new grey level using modified membership and non-membership values

$$g'_{mn} = g_{max} - F_d * \left(\sqrt{\mu'_{mn}(C_1 - \nu'_{mn})}\right)^{\frac{-1}{C_2 * F_e}} + F_d$$

where C_1 and C_2 are arbitrary constants.

(vii) Display IF filtered image.

4.8 Results and Discussion

The proposed IF algorithm, find better performance to handle imprecision in grey distribution. Four types of IF filters namely IF mean, IF median, IF maximum, IF minimum are taken for analysis. A grey scale cameraman image of size 256 x 256 with 8 bits per pixel tone resolution with various noise is considered for analysis and is subjected to different types of noise namely, salt and pepper noise, gaussian noise, poisson noise and speckle noise are tested against various IF filters. IF filter of window size 3 x 3 is applied to the grey scale image. The performance shows the results of IF filter against various noise types. Comparison based on image performance is displayed in Table 4.8 and Table 4.9.

Correlation Coefficient (CC) method is widely used for comparing two images and for measuring the association between two images. Hence, it is used to determine how close the input and output images co-vary. The Correlation Coefficient is defined as

$$r = \frac{\sum\limits_{i=1}^{n}(X_i - X)(Y_i - Y)}{\sqrt{\sum\limits_{i=1}^{n}(X_i - X)^2}\sqrt{\sum\limits_{i=1}^{n}(Y_i - Y)^2}} \tag{4.1}$$

where X_i is the intensity of the i^{th}pixel in the original image, Y_i is the intensity of the i^{th} pixel in the restored image, X is the mean intensity of the original image and Y is the mean intensity of the restored image.

Pearson's coefficient varies from -1 to 1. Low absolute values of r means that two variables posses weak association (uncorrelated). If $r = 1$ or (-1), it is positively (negatively) correlated. For qualitative analysis, performances of the filters are tested at different level of noise densities, and results are tabulated. The performance of the resulting output image is quantified using PSNR and MSE values. Restored image performance using PSNR is defined as below:

$$PSNR = 10log_{10} \frac{255^2}{\frac{1}{mn}\sum_{i,j}(r_{i,j} - x_{i,j})^2} \tag{4.2}$$

$$MSE = \frac{1}{mn}\sum_{i,j}(r_{i,j} - x_{i,j})^2 \tag{4.3}$$

where r - original image, x - restored image and mn - size of the image.

Different types of noise	**1**	**2**	**3**	**4**
Salt and pepper noise	0.9678	0.9822	0.6896	0.7839
Gaussian noise	0.9479	0.9418	0.8729	0.8240
Poisson noise	0.9754	0.9793	0.9056	0.9040
Speckle noise	0.9703	0.9662	0.9134	0.9087

Table 4.8: Correlation Coefficient

1 - CC of IF mean filter; 2 - CC of IF median filter; 3 - CC of IF maximum filter; 4 - CC of IF minimum filter.

CC for various noise levels are calculated and are listed in Table 4.8. The higher values of CC infer that output of the proposed algorithm is in acceptable ratio.

Cameraman image	**IF mean filter**		**IF med filter**		**IF max filter**		**IF min filter**	
Types of noise	PSNR	MSE	PSNR	MSE	PSNR	MSE	PSNR	MSE
Salt & pepper	56.91	0.1324	58.62	0.0821	55.995	0.1636	57.44	0.1172
Gaussian	56.56	0.1433	56.75	0.1374	54.97	0.2070	57.82	0.1074
Poisson	57.16	0.1249	57.06	0.1273	56.26	0.1535	57.59	0.1131
Speckle	57.81	0.1075	57.93	0.1047	56.82	0.90876	58.36	0.0947

Table 4.9: PSNR and MSE values for various types of noise

PSNR and MSE values for the various noise tested against IF filters are tabulated in Table 4.9. The higher values of PSNR and lower values of MSE infer that output of the proposed algorithm is in acceptable ratio.

Chapter 5

Intuitionistic fuzzy moving average in decision making

The current topic is to investigate the multi-period decision making problem where the decision information are provided by decision maker at different periods and are represented in the form of intuitionistic fuzzy index matrix. It further analyzes the use of IF moving averages with aggregation operators, distance measures and OWA operators. A real example, to show the effectiveness of the proposed algorithm, is carried out to employ growth of Gross Domestic Product (GDP) of Indian Economy in practical applications. A comparitive analysis has been made among crisp, fuzzy and IF moving aggregation operators. An IF multi-period decision making algorithm is designed using IF weighted moving average. Ranking is performed on the basis of the score function. Finally, it is inferred that, IF weighted moving average is the most influencing factor to the growth of GDP.

5.1 Basic definitions

In this section, the basic notions, concepts and definitions necessary for the study are briefly reviewed.

Definition 5.1.1. [27]

For any two IFSs A and B in E, the operator @ is defined as

$$A@B = \left\{ \left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} \right\rangle : x \in E \right\}$$

Definition 5.1.2. [9]

Let $\alpha, \beta \in [0, 1]$. Given an IFS A, the operator $G_{\alpha,\beta}(A)$ is defined as

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha \cdot \mu_A(x), \beta \cdot \nu_A(x)\rangle : x \in E\}$$

Definition 5.1.3. [106]

An *ordered weighted averaging operator* of dimension n is a mapping $OWA : R^n \to R$ that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, Furthermore,

$$OWA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j b_j$$

where b_j is the j^{th} largest of a_i $(i = 1, 2, ..., n)$.

Definition 5.1.4. [21]

The *intuitionistic fuzzy pair* (IFP) is an ordered pair of the form $\langle a, b\rangle$, where $a, b \in [0,1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process, and which components (a and b) are interpreted, respectively, as degrees of membership and non-membership to a given set.

Definition 5.1.5. [97]

For any two IFSs A and B in $X = \{x_1, x_2, \cdots, x_n\}$, the Hamming distance is defined as

$$d_H(A, B) = \frac{1}{2}\sum_{j=1}^{n}(|\mu_A(x_j) - \mu_B(x_j)| + |\nu_A(x_j) - \nu_B(x_j)| + |\pi_A(x_j) - \pi_B(x_j)|)$$

Definition 5.1.6. [41]

Let $A(\mu, \nu)$ be an IFS. Then $S(A) = \frac{(\mu+1-\nu)}{2}$ is called as the *score* of A where $S(A) \in [0,1]$.

Definition 5.1.7. [9]

Let $A(\mu, \nu)$ be an IFS. Then $L(A) = \mu + \nu$ be the *accuracy* of A where $L(A) \in [0,1]$.

Definition 5.1.8. [9]

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{R}$ be the set of the real numbers. *Intuitionistic*

Fuzzy Index Matrix with index sets K and L ($K, L \subset \mathcal{I}$), takes the form:

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \langle \mu_{k_1,l_2}, \nu_{k_1,l_2} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ k_2 & \langle \mu_{k_2,l_1}, \nu_{k_2,l_1} \rangle & \langle \mu_{k_2,l_2}, \nu_{k_2,l_2} \rangle & \dots & \langle \mu_{k_2,l_n}, \nu_{k_2,l_n} \rangle \\ \vdots & \vdots & \vdots & \dots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \langle \mu_{k_m,l_2}, \nu_{k_m,l_2} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, for $1 \leq i \leq m$,and $1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$ i.e.,$\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$ is an IF pair.

5.1.1 Moving average

Moving average is a succession of averages that aggregates the subset of successive segments in a bigger set of information and moves towards some part of the whole set which is available or to be obtained in the future [60]. The moving average is usually seen as moving aggregation operator and can be used in aggregation problems including multi-period aggregation and multi-criteria aggregation. In literature, different types of moving averages are assessed, in which arithmetic moving average and the weighted moving average are considered important.

Definition 5.1.9. [60]

An *arithmetic moving average* of dimension m is a mapping $MA : R^m \to R$ such that

$$MA(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \sum_{i=1+p}^{m+p} a_i$$

where a_i is the i^{th} argument and m is the total number of arguments considered from the whole sample, p indicates the movement done in the average from the initial position.

Definition 5.1.10. [60]

A *Weighted Moving Average* (WMA) of dimension m is a mapping $WMA : R^m \to R$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0, 1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$WMA(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \sum_{j=1+p}^{m+p} w_j a_j,$$

where m is the total number of arguments considered from the whole sample, p indicates the movement done in the average from the initial position.

Note

If $p = 0$, first argument of the sample is considered for calculating the moving average, if $p = 1$ move an average one position by each from the initial position considering first the second argument, if $p = 3$ move an average 3 positions from initial position, considering first the fourth argument and so on.

Definition 5.1.11. [64]

Fuzzy weighted moving average (F_{WMA}) of dimension m is a mapping $F_{WMA} : R^m \to R$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0, 1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$F_{WMA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \sum_{j=1+p}^{m+p} w_j a_j,$$

where m is the total number of arguments considered from the whole sample, p indicates the movement done in the average from the initial position.

5.1.2 Moving Distance Measure

Moving distance measure is a distance measure that move towards the whole sample and are represented in a dynamic way [60]. An aggregation process is found by using weighted averages resulting in weighted moving averaging distance (WMAD) [102]. Its main advantage is that weights are associated to each distance depending on the relevancy of each distance in the analysis. For two sets, $X = \{x_{1+p}, x_{2+p}, ..., x_{m+p}\}$ and $Y = \{y_{1+p}, y_{2+p}, ..., y_{m+p}\}$, WMAD can be defined as follows.

Definition 5.1.12. [60]

A $WMAD$ of dimension m is a mapping $WMAD : R^m \times R^m \to R$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_i \in [0, 1]$ and $\sum_{i=1+p}^{m+p} w_i = 1$ such that

$$WMAD((x_{1+p}, y_{1+p}), (x_{2+p}, y_{2+p}), ..., (x_{m+p}, y_{m+p})) = \sum_{i=1+p}^{m+p} w_i d(x_i, y_i)$$

where x_i and y_i are the i^{th} arguments of the sets X and Y respectively, $d(x_i, y_i)$ is the averaging distance between x_i and y_i, m is the total number of arguments considered from the whole sample, and p indicates the movement done in the average from the initial position.

5.2 Intuitionistic fuzzy averaging operators

Let $\wp = \{\langle\mu,\nu\rangle : \mu,\nu \in [0,1], 0 \leq \mu+\nu \leq 1\}$ be a set of all intuitionistic fuzzy pairs,

$\wp^m = \{(a_1, a_2, ..., a_m)|a_j = \langle\mu_j,\nu_j\rangle, j = 1, 2, ..., m, \mu_j, \nu_j \in [0,1], 0 \leq \mu_j + \nu_j \leq 1\}$

be the set of all m dimensional intuitionistic fuzzy pairs and m is the sample size.

5.2.1 Intuitionistic fuzzy ordered weighted averaging operator

Definition 5.2.1.

Intuitionistic fuzzy ordered weighted averaging operator of dimension m is a mapping $IF_{OWA} : \wp^m \to \wp$ such that

$$IF_{OWA}(a_1, a_2, ..., a_m) = \mathop{@}_{j=1}^{m} G_{\alpha,\beta}(a_j),$$

where $\alpha, \beta \in [0,1]$.

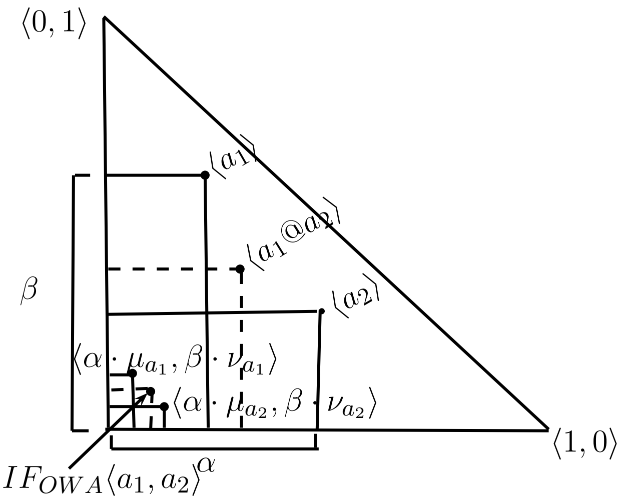


Figure 5.1: Geometric interpretation of the operator IF_{OWA}

Let a_1 and a_2 be two intuitionistic fuzzy pairs which assign a point $IF_{OWA}(a_1, a_2)$ with coordinates $\langle \alpha \cdot \frac{\mu(a_1)+\mu(a_2)}{2}, \beta \cdot \frac{\nu(a_1)+\nu(a_2)}{2} \rangle$ depending on the value of the arguments $\alpha, \beta \in [0, 1]$. The geometric interpretation of the operator IF_{OWA} is shown in Figure 5.1.

5.2.2 Intuitionistic fuzzy moving average

Intuitionistic fuzzy moving average is the mean of IF data for several consecutive time periods where the observations are equally spaced. It is an indicator that reacts to events that have already happened and used as an interpretive for confirmations and analysis.

Definition 5.2.2.

An *intuitionistic fuzzy moving average* (IF_{MA}) is a mapping $IF_{MA} : \wp^m \to \wp$

such that

$$IF_{MA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \mathop{@}_{j=1+p}^{m+p} a_j = \left\langle \frac{1}{m} \sum_{j=1+p}^{m+p} \mu_j, \frac{1}{m} \sum_{j=1+p}^{m+p} \nu_j \right\rangle.$$

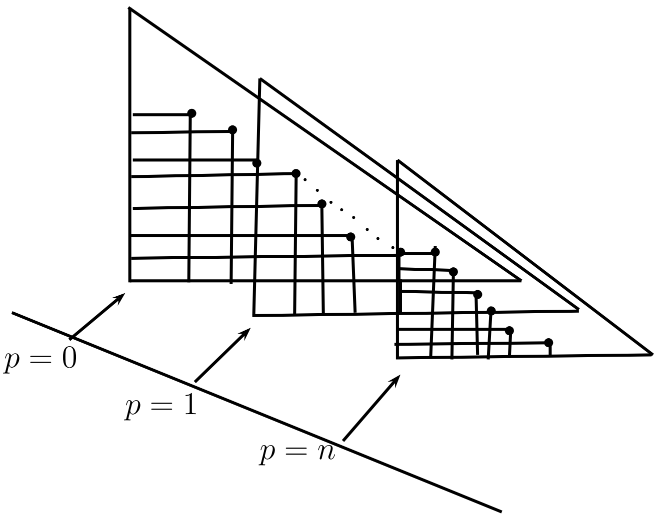


Figure 5.2: Geometric interpretation of the operator IF_{MA}

Let $a_1, a_2, ..., a_m$ be the collection of intuitionistic fuzzy pairs which assign a point $IF_{MA}(a_{1+p}, a_{2+p}, ..., a_{m+p})$ for every p, where p indicates the movement done in the average from the initial position. The geometric interpretation of the operator IF_{MA} is exhibited in Figure 5.2. An extract of the operator IF_{MA} is displayed in Figure 5.3.

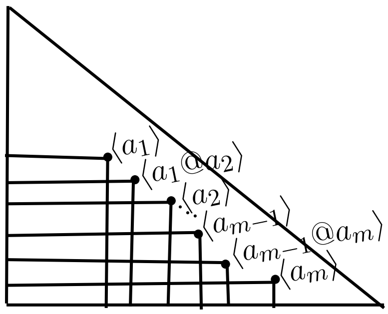


Figure 5.3: An extract of the operator IF_{MA}

5.3 Types of IF moving average

In this section, six types of intuitionistic fuzzy moving averages are discussed.

Definition 5.3.1.

An *intuitionistic fuzzy weighted moving average* (IF_{WMA}) of dimension m is a mapping $IF_{WMA} : \wp^m \to \wp$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0, 1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$IF_{WMA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \sum_{j=1+p}^{m+p} w_j a_j = \left\langle 1 - \prod_{j=1+p}^{m+p} (1-\mu_j)^{w_j}, \prod_{j=1+p}^{m+p} \nu_j^{w_j} \right\rangle$$

Intuitionistic fuzzy ordered weighted moving average

The prominent characteristics of intuitionistic fuzzy ordered weighted moving average is the reordering step, where it first reorders all the given arguments in descending order and then weights these ordered arguments, and finally aggregates all these ordered weighted arguments into a collective one.

Definition 5.3.2.

An *intuitionistic fuzzy ordered weighted moving average* (IF_{OWMA}) of dimension m is a mapping $IF_{OWMA} : \wp^m \rightarrow \wp$ that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_k \in [0, 1]$ and $\sum_{k=1+p}^{m+p} w_k = 1$ such that

$$IF_{OWMA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \sum_{k=1+p}^{m+p} w_k b_k = \left\langle 1 - \prod_{k=1+p}^{m+p} (1 - \mu_k)^{w_k}, \prod_{k=1+p}^{m+p} \nu_k^{w_k} \right\rangle,$$

where $b_k = \langle \mu_k, \nu_k \rangle$ is the k^{th} largest IF pair of a_j which is determined using intuitionistic fuzzy ranking methods [57].

Definition 5.3.3.

An *intuitionistic fuzzy ordered weighted averaging - weighted moving average* of dimension m is a mapping $IF_{OWAWMA} : \wp^m \rightarrow \wp$ that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0, 1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ and a weighting vector V with $\sum_{i=1+p}^{m+p} v_i = 1$ and $v_i \in [0, 1]$ such that

$$IF_{OWAWMA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \lambda \sum_{j=1+p}^{m+p} w_j b_j + (1 - \lambda) \sum_{i=1+p}^{m+p} v_i a_i$$

where $b_j = \langle \mu_j, \nu_j \rangle$ is the j^{th} largest IF value of a_i, $\lambda \in [0, 1]$ and a_i are the argument variables represented in the form of IF pairs.

Note

IF_{OWAWMA} is a new model that unifies both IF_{OWMA} operator and IF_{WMA} operator considering the degree of importance that each one has in the aggregation process.

Definition 5.3.4.

An *intuitionistic fuzzy induced ordered weighted moving average* (IF_{IOWMA}) of dimension m is a mapping $IF_{IOWMA} : R^m \times \wp^m \to \wp$ that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0, 1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$IF_{IOWMA}(\langle u_{1+p}, a_{1+p}\rangle, \langle u_{2+p}, a_{2+p}\rangle, ..., \langle u_{m+p}, a_{m+p}\rangle) = \sum_{j=1+p}^{m+p} w_j b_j$$

$$= \left\langle 1 - \prod_{j=1+p}^{m+p} (1 - \mu_j)^{w_j}, \prod_{j=1+p}^{m+p} \nu_j^{w_j} \right\rangle,$$

where b_j is the a_i value of IF_{IOWMA} pair $\langle \mu_i, a_i\rangle$ having j^{th} largest u_i and u_i is the order inducing variable and $a_i = \langle \mu_i, \nu_i\rangle$ is an IF pair.

Note

The main difference between IF_{IOWMA} and IF_{OWMA} is the reordering step in IF_{IOWMA}. It is not carried out with the values of the arguments, but with order inducing variables which changes due to dynamic situations in the imprecise environment. Auxiliary variable associated with each input is taken as an inducing variable.

Definition 5.3.5.

An *intuitionistic fuzzy weighted geometric moving average* (IF_{WGMA}) of dimension m is a mapping $IF_{WGMA} : \wp^m \to \wp$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0, 1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$IF_{WGMA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \sum_{j=1+p}^{m+p} a_j{}^{w_j} = \left\langle \prod_{j=1+p}^{m+p} (\mu_j)^{w_j}, 1 - \prod_{j=1+p}^{m+p} (1-\nu_j)^{w_j} \right\rangle$$

Definition 5.3.6.

An *intuitionistic fuzzy weighted harmonic moving average* (IF_{WHMA}) of dimension m is a mapping $IF_{WHMA} : \wp^m \to \wp$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0,1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$IF_{WHMA}(a_{1+p}, a_{2+p}, ..., a_{m+p}) = \frac{1}{\sum\limits_{j=1+p}^{m+p} \frac{w_j}{a_j}} = \left\langle \frac{1}{\sum\limits_{j=1+p}^{m+p} \frac{w_j}{\mu_j}}, 1 - \frac{1}{\sum\limits_{j=1+p}^{m+p} \frac{w_j}{1-\nu_j}} \right\rangle$$

Numerical Example

Example 5.3.1. Consider the set of closing prices of the stock values from Eastman Kodak for 10 days. Track the closing price of the stock values to smooth the data.

Assume $W = (0.5, 0.2, 0.3)$ and the order inducing variable of IF_{IOWMA} is $U = \{12, 14, 21\}$. Let A be the closing price of the stock values for 10 days. Here, $iftrif$ is used for fuzzification process. Let $a = 2379$, $b = 2800$, $c = 3200$ and $\epsilon = 0.1$

DAYS	1	2	3	4	5	6	7	8	9	10
A	2880	3039	3075	2899	2988.5	2945	2879	2800	2887	3100

Table 5.1: Set of closing price of the stock values for 10 days (in INR)

IF pair of A	IF_{MA}		IF_{WMA}		IF_{OWMA}		IF_{IOWMA}	
	IF pair	**score**	**IF pair**	**score**	**IF pair**	**score**	**IF pair**	**score**
$\langle 0.7, 0.2\rangle$								
$\langle 0.3025, 0.5975\rangle$	$\langle 0.405, 0.495\rangle$	0.455	$\langle 0.5256, 0.3605\rangle$	0.5825	$\langle 0.5256, 0.3605\rangle$	0.5825	$\langle 0.4245, 0.4615\rangle$	0.4815
$\langle 0.2125, 0.6875\rangle$	$\langle 0.3891, 0.5108\rangle$	0.4391	$\langle 0.4201, 0.4717\rangle$	0.4742	$\langle 0.4894, 0.4010\rangle$	0.5442	$\langle 0.4955, 0.3954\rangle$	0.5500
$\langle 0.6525, 0.2475\rangle$	$\langle 0.4312, 0.4687\rangle$	0.4812	$\langle 0.3927, 0.5003\rangle$	0.4462	$\langle 0.5094, 0.3824\rangle$	0.5635	$\langle 0.4305, 0.4639\rangle$	0.4833
$\langle 0.4287, 0.4712\rangle$	$\langle 0.5395, 0.3604\rangle$	0.5895	$\langle 0.5818, 0.3156\rangle$	0.6331	$\langle 0.5728, 0.3240\rangle$	0.6244	$\langle 0.5572, 0.3406\rangle$	0.6083
$\langle 0.5375, 0.3625\rangle$	$\langle 0.5562, 0.3437\rangle$	0.6062	$\langle 0.5497, 0.3444\rangle$	0.6026	$\langle 0.6048, 0.2844\rangle$	0.6602	$\langle 0.6048, 0.2894\rangle$	0.6577
$\langle 0.7025, 0.1975\rangle$	$\langle 0.7133, 0.1866\rangle$	0.7633	$\langle 0.7325, 0\rangle$	0.8662	$\langle 0.8031, 0\rangle$	0.9015	$\langle 0.8031, 0\rangle$	0.9015
$\langle 0.9, 0\rangle$	$\langle 0.7616, 0.1383\rangle$	0.8116	$\langle 0.7654, 0\rangle$	0.8827	$\langle 0.8241, 0\rangle$	0.9120	$\langle 0.7528, 0\rangle$	0.8764
$\langle 0.6825, 0.2175\rangle$	$\langle 0.5775, 0.3225\rangle$	0.6275	$\langle 0.7605, 0\rangle$	0.8802	$\langle 0.7605, 0\rangle$	0.8802	$\langle 0.6326, 0\rangle$	0.8163
$\langle 0.15, 0.75\rangle$								

Table 5.2: Different types of IF moving averages (3-day moving)

Four types of 3−day intuitionistic fuzzy moving averages are calculated and listed in Table 5.2. In column 2, IF moving average gives the average price of stock values for 10 days. At the end of every new day, the oldest data point is dropped and the newest one is added to the beginning. In column 3, the *score* values are increasing from 0.455 to 0.6275, showing increase in trend. In column 4, $IF_{WMA}$is calculated. Each point within the period is assigned a multiplier which changes the weight or significance of a particular data point.

In column 5, the *score* values are increasing from 0.5825 to 0.8802, showing increasing trend. In columns 6 and 7, IF_{OWMA} and IF_{IOWMA} are calculated respectively. As values increase they all show an increasing trend. Hence, it is inferred, from the entries of Table 5.2, that closing prices over a specified time period is increasing.

5.4 Intuitionistic fuzzy moving distance measure

Distance measures are used for measuring the deviations of different arguments [60]. In the existing literature, a variety of IF distance measures have been introduced. IF distance measures are important in various scientific fields such as decision making, pattern recognition, machine learning and market prediction. Here, IF distance measures are extended using moving average operators resulting in IF moving distance measures.

Intuitionistic fuzzy moving distance makes the comparison between two sets of elements in a dynamic way to assess the information at different time periods. It moves towards a sample considering different partial aggregations that can be obtained by using different information from the sample. IF moving distance aggregation operators can be used in multi-period aggregation and multi-criteria aggregation problems.

Notations

Let $X = \{x_{1+p}, x_{2+p}, ..., x_{m+p}\}$ and $Y = \{y_{1+p}, y_{2+p}, ..., y_{m+p}\}$ be two IF sets, m is the sample size and p indicates the movement done in the average from the initial position.

Definition 5.4.1.

An *intuitionistic fuzzy moving average distance* (IF_{MAD}) is a mapping IF_{MAD} : $\wp^m \times \wp^m \to R$ such that

$$IF_{MAD}((x_{1+p}, y_{1+p}), (x_{2+p}, y_{2+p}), ..., (x_{m+p}, y_{m+p})) = \sum_{i=1+p}^{m+p} d(x_i, y_i)$$
$$= \frac{1}{2}\sum_{i=1+p}^{m+p}(|\mu(x_i) - \mu(y_i)| + |\nu(x_i) - \nu(y_i)| + |\pi(x_i) - \pi(y_i)|)$$

where x_i and y_i are the i^{th} argument of the sets X and Y respectively, $d(x_i, y_i)$ is the IF averaging distance between x_i and y_i.

Definition 5.4.2.

An *intuitionistic fuzzy weighted moving average distance* (IF_{WMAD}) of dimension m is a mapping IF_{WMAD} : $\wp^m \times \wp^m \to R$ with an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_i \in [0, 1]$ and $\sum_{i=1+p}^{m+p} w_i = 1$ such that

$$IF_{WMAD}((x_{1+p}, y_{1+p}), (x_{2+p}, y_{2+p}), ..., (x_{m+p}, y_{m+p})) = \sum_{i=1+p}^{m+p} w_i d(x_i, y_i)$$
$$= \frac{1}{2}\sum_{i=1+p}^{m+p} w_i(|\mu(x_i) - \mu(y_i)| + |\nu(x_i) - \nu(y_i)| + |\pi(x_i) - \pi(y_i)|)$$

where x_i and y_i are the i^{th} argument of the sets X and Y respectively, $d(x_i, y_i)$ is the IF averaging distance between x_i and y_i.

Intuitionistic fuzzy ordered weighted moving average distance

Motivated by the idea of the OWA operator, Zeng developed an OWD measure in intuitioinistic fuzzy environment [111]. In this section, OWD is further extended using intuitionistic fuzzy moving average which add weightage to the ordered position of each deviation value. IF_{OWMAD} provides a family of distance aggregation operators between IFSs.

Definition 5.4.3.

An *intuitionistic fuzzy ordered weighted moving average distance* (IF_{OWMAD}) of dimension m is a mapping $IF_{OWMAD} : \wp^m \times \wp^m \to R$ that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_k \in [0,1]$ and $\sum_{k=1+p}^{m+p} w_k = 1$ such that

$$IF_{OWMAD}((x_{1+p}, y_{1+p}), (x_{2+p}, y_{2+p}), ..., (x_{m+p}, y_{m+p})) = \sum_{k=1+p}^{m+p} w_k d(x_k, y_k)$$
$$= \frac{1}{2} \sum_{k=1+p}^{m+p} w_k (|\mu(x_k) - \mu(y_k)| + |\nu(x_k) - \nu(y_k)| + |\pi(x_k) - \pi(y_k)|)$$

where $d(x_k, y_k)$ is the IF ordered weighted averaging distance between the k^{th} largest arguments x_k and y_k of the sets X and Y represented in the form of individual distance.

Definition 5.4.4.

An *intuitionistic fuzzy induced ordered weighted moving average distance* (IF_{IOWMAD}) of dimension m is a mapping $IF_{IOWMAD} : R^m \times \wp^m \times \wp^m \to R$ that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0,1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$IF_{IOWMAD}(\langle u_{1+p}, x_{1+p}, y_{1+p}\rangle, \langle u_{2+p}, x_{2+p}, y_{2+p}\rangle, ..., \langle u_{m+p}, x_{m+p}, y_{m+p}\rangle)$$
$$= \sum_{j=1+p}^{m+p} w_j d(x_j, y_j)$$
$$= \frac{1}{2} \sum_{j=1+p}^{m+p} w_j (|\mu(x_j) - \mu(y_j)| + |\nu(x_j) - \nu(y_j)| + |\pi(x_j) - \pi(y_j)|)$$

where $d(x_j, y_j)$ is the IF induced ordered weighted averaging distance value of the triplet $\langle \mu_i, x_i, y_i \rangle$ having j^{th} largest u_i, u_i is the ordered inducing variable, $d(x_i, y_i)$ is the argument variable represented in the form of individual distance.

The validity of the formulae defined in this section are verified in Example 5.4.1.

Example 5.4.1. A chemical sales company monitors inventory levels (A) for two of its chemical products A and B each day. Data are recorded for 10 days. Let X= sales price of the chemical product A, Y= sales price of the chemical product B. Assume that the weighting vector is $W = (0.4, 0.5, 0.1)$. Compare the two variables X and Y and formulate their differences using intuitionistic fuzzy moving distance operators. Here, $iftrif$ is used for fuzzification process.

Days	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{X}$	161	99	135	120	164	221	179	204	214	101
$\boldsymbol{Y}$	157.27	93.28	136.81	123.79	153.51	241.74	201.54	206.71	229.78	135.22

Table 5.3: Set of arguments for 10 days (in INR)

Three types of 3-day intuitionistic fuzzy moving average distances are calculated and listed in Table 5.4. In column 3, IF_{MAD} computes difference in the average price of chemical sales. The distance values are increasing from 0.0852 to 0.7643, which infer decrease in price value. Similarly IF_{WMAD}, IF_{OWMAD} are calculated in columns 4 and 5 respectively and the distance values are increasing. Hence, from Table 5.4, it is seen that on comparing the average sales of the products, there is a decrease in sales.

$\boldsymbol{X}$	$\boldsymbol{Y}$	$\boldsymbol{IF_{MAD}}$	$\boldsymbol{IF_{WMAD}}$	$\boldsymbol{IF_{OWMAD}}$
$\langle 0.9438, 0.0462\rangle$	$\langle 0.8865, 0.1035\rangle$			
$\langle 0.0000, 0.9900\rangle$	$\langle 0.0000, 0.9900\rangle$	0.0852	0.0257	0.0257
$\langle 0.5438, 0.4462\rangle$	$\langle 0.5717, 0.4183\rangle$	0.0862	0.0197	0.0372
$\langle 0.3131, 0.6769\rangle$	$\langle 0.3714, 0.6186\rangle$	0.2475	0.05643	0.0964
$\langle 0.9900, 0.0000\rangle$	$\langle 0.8286, 0.1614\rangle$	0.4765	0.1297	0.1892
$\langle 0.2568, 0.7332\rangle$	$\langle 0.0000, 0.9900\rangle$	0.7081	0.2219	0.2605
$\langle 0.7970, 0.1930\rangle$	$\langle 0.5071, 0.4829\rangle$	0.5816	0.2571	0.2478
$\langle 0.4755, 0.5145\rangle$	$\langle 0.4406, 0.5494\rangle$	0.5278	0.1537	0.2209
$\langle 0.3468, 0.6432\rangle$	$\langle 0.1438, 0.8462\rangle$	0.7643	0.1680	0.3155
$\langle 0.0208, 0.9692\rangle$	$\langle 0.5472, 0.4428\rangle$			

Table 5.4: Different types of IF moving average distances (3-day moving)

5.5 Multi-period decision making using intuitionistic fuzzy averaging operator

Multi-period decision making is the process of finding the best alternative for k-periods from all of the feasible alternatives where all the alternatives can be evaluated according to a number of attributes. In general, multi-period decision making problem contains uncertain and imprecise information. Suppose that there are n alternatives $x_j, j = 1, 2, ..., n$ evaluated with respect to m attributes $O_i, i = 1, 2, ..., m$. The set of alternatives and attributes are denoted by $X = \{x_1, x_2, ..., x_n\}$ and $O = \{o_1, o_2, ..., o_m\}$ respectively. Evaluation of any alternative $x_j \in X$ on each attribute $o_i \in O$ is expressed with an intuitionistic fuzzy set for multi-period as ${F_{ij}}^k = \{\langle (o_i, x_j), \mu_{ij}, \nu_{ij}\rangle\}$, where $\mu_{ij} \in [0, 1]$ and $\nu_{ij} \in [0, 1]$ such that $0 \leq \mu_{ij} + \nu_{ij} \leq 1$, denote ${F_{ij}}^k\{\langle (o_i, x_j), \mu_{ij}, \nu_{ij}\rangle\}$ by ${F_{ij}}^k = \langle \mu_{ij}, \nu_{ij}\rangle$ [56].

Intuitionistic fuzzy index matrix for k-multi-period decision making is expressed as follows [11] :

$$F^k = \begin{array}{c} \\ o_1 \\ o_2 \\ \ldots \\ o_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \ldots & x_n \end{array} \\ \begin{pmatrix} \langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \ldots & \langle \mu_{1n}, \nu_{1n} \rangle \\ \langle \mu_{21}, \nu_{21} \rangle & \langle \mu_{22}, \nu_{22} \rangle & \ldots & \langle \mu_{2n}, \nu_{2n} \rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle \mu_{m1}, \nu_{m1} \rangle & \langle \mu_{m2}, \nu_{m2} \rangle & \ldots & \langle \mu_{mn}, \nu_{mn} \rangle \end{pmatrix} \end{array}$$

Attributes may be of different importance. Assume that the weight of each attribute $o_i \in O$ is W, which satisfy the normalized conditions: $w_i \in [0, 1](i = 1, 2, ..., m)$ and $\sum_{i=1}^{m} w_i = 1$.

Algorithm for multi-period decision making using intuitionistic fuzzy weighted average:

Based on the intuitionistic fuzzy weighted averaging operator, algorithm for multi-period decision making under IF environment is proposed, which involves the following steps:

Step 1: Define the universe of discourse, $U = [D_{min}, D_{max}]$, based on the range of available economical time series data, where D_{min}, D_{max} are minimum and maximum of time series data, respectively and treat the attributes as linguistic variables on U.

Step 2: Describe the linguistical terms.

Step 3: Form the membership function and non-membership function for each linguistic term of the identified attributes. That is, construct an IF set A_i and apply *iftrif* to fuzzify the crisp data.

Step 4: Evaluate each alternative $x_j \in X$ and aggregate the information using

IF_{WMA} operator, that has an associated weighting vector $W = (w_1, w_2, ..., w_m)^T$, $w_j \in [0,1]$ and $\sum_{j=1+p}^{m+p} w_j = 1$ such that

$$IF_{WMA}(F^{1+p}, F^{2+p}, ..., F^{m+p}) = \sum_{j=1+p}^{m+p} w_j F^j = \left\langle 1 - \prod_{j=1+p}^{m+p} (1-\mu_j)^{w_j}, \prod_{j=1+p}^{m+p} \nu_j^{w_j} \right\rangle,$$

where m is the total number of arguments considered from the whole sample, p indicates the movement done in the average from the initial position.

Step 5: Calculate the scores.

Step 6: Rank and select the best alternative.

Step 7: End

5.6 Numerical example

Gross Domestic Product

GDP is the final value of the goods and services produced within the geographic boundaries of a country during a specified period of time, normally a year. Its growth rate is an indicator of the economic performance of a country and it measures the nation's total output of goods and services useful for a wide variety of purposes such as measuring productivity, conducting monetary policy, and projecting tax revenues and provides the information about the size of the economy and how an economy is performing.

Indian Economy is classified into three sectors namely Agriculture & allied, Industry and Services. Agriculture sector includes agriculture (agriculture proper and livestock), forestry & logging, fishing and related activities. Industry includes Mining & quarrying, manufacturing, electricity, gas, water supply, and construction.

Services sector is comprised of trade, repair, hotels and restaurants, transport, storage, communication & services related to broadcasting, financial, real estate, community and social services, public administration, defence and other services.

Example 5.6.1. A real time example considering GDP growth of Indian Economy is taken for analysis and IF_{WMA} operator is used to find application in multi-period decision making.

Year	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$	$\boldsymbol{A_3}$	$\boldsymbol{A_4}$	$\boldsymbol{A_5}$	$\boldsymbol{A_6}$
2010	14.88	15.12	13.03	13.93	12.69	16.41
2009	12.75	12.62	12.78	12.03	11.68	18.42
2008	15.7	18.46	16.67	16.88	15.42	15.56
2007	13.36	12.64	19.53	13.05	21.69	16.19
2006	12.79	12.63	14.95	11.09	15.1	14.12

Table 5.5: GDP for 5 years

GDP of Indian Economy is taken for five years. It comprises of six major sectors Agriculture & allied, Agriculture, Industry, Mining & quarrying, Manufacturing and Service sector. Consider the information from 2006 to 2010 in order to provide an appropriate forecast that permits to reach the optimal decision. Evaluate each alternative analyzing the benefits they could give each year. Find the sector which influences the growth of GDP in the forth coming year out of six sectors. Actual GDP data [117] for the period 2010 - 2006 are shown in Table 5.5.

Step 1:

Let the universe of discourse be $U = [11.09, 21.69]$. Consider the six attributes $A_1, A_2, ..., A_6$ as linguistic variables on U as follows:

A_1 : Contribution of Agriculture & allied sector

A_2 : Contribution of Agriculture sector

A_3 : Contribution of Industry sector

A_4 : Contribution of Mining & quarrying sector

A_5 : Contribution of Manufacturing sector

A_6 : Contribution of Service sector

Step 2:

Describe the linguistics terms. The alternatives are

R : Economic recession, S : Stable economy, B : Boom economic condition

Step 3:

In order to deal imprecise and uncertain information in the available data, intuitionistic fuzzifiaction is essential to model the data. Construct an IF set and apply *iftrif* to fuzzify the crisp data. That is, form the membership function and non-membership function for each linguistic term of the identified attributes. The degree of membership is represented by μ, degree of non-membership is represented by ν and degree of uncertainty is represented by ϵ, an arbitrary parameter chosen in such a way that $\mu_A(x) + \nu_A(x) + \epsilon = 1$ and $0 \leq \epsilon < 1$.

Assume $a = 11.09$, $b = 15.7$, $c = 21.69$ and $\epsilon = 0.01$.

The membership and non-membership functions for *Recession* are defined as

$$\mu_R A(x) = \begin{cases} 1-\epsilon & ; \quad x \leq a \\ (\frac{b-x}{b-a}) - \epsilon & ; \quad a < x < b \\ 0 & ; \quad x \geq b \end{cases}$$

and

$$\nu_R A(x) = \begin{cases} 0 & ; \quad x \leq a \\ 1 - (\frac{x-a}{b-a}) & ; \quad a < x < b \\ 1-\epsilon & ; \quad x \geq b \end{cases}$$

The membership and non-membership functions for *Stable* are as follows

$$\mu_S A(x) = \begin{cases} 0 & ; \quad x \leq a \\ (\frac{x-a}{b-a}) - \epsilon & ; \quad a < x \leq b \\ (\frac{c-x}{c-b}) - \epsilon & ; \quad b \leq x < c \\ 0 & ; \quad x \geq c \end{cases}$$

and

$$\nu_S A(x) = \begin{cases} 1-\epsilon & ; \quad x \leq a \\ 1 - (\frac{x-a}{b-a}) & ; \quad a < x \leq b \\ 1 - (\frac{c-x}{c-b}) & ; \quad b \leq x < c \\ 1-\epsilon & ; \quad x \geq c \end{cases}$$

The membership and non-membership functions of *Boom* takes the form

$$\mu_B A(x) = \begin{cases} 0 & ; \quad x \leq a \\ (\frac{x-a}{b-a}) - \epsilon & ; \quad a < x < b \\ 1 - \epsilon & ; \quad x \geq b \end{cases}$$

and

$$\nu_B A(x) = \begin{cases} 1 - \epsilon & ; \quad x \leq a \\ 1 - (\frac{x-a}{b-a}) & ; \quad a < x < b \\ 0 & ; \quad x \geq b \end{cases}$$

The fuzzified economical time series data of GDP are displayed in Tables 5.6 - 5.10

	R	**S**	**B**
$\boldsymbol{A_1}$	$\langle 0.63185, 0.358155\rangle$	$\langle 0.812125, 0.17788\rangle$	$\langle 0.34721, 0.64279\rangle$
$\boldsymbol{A_2}$	$\langle 0.60923, 0.380773\rangle$	$\langle 0.86419, 0.12581\rangle$	$\langle 0.36983, 0.62017\rangle$
$\boldsymbol{A_3}$	$\langle 0.80621, 0.18379\rangle$	$\langle 0.41082, 0.57918\rangle$	$\langle 0.17285, 0.817155\rangle$
$\boldsymbol{A_4}$	$\langle 0.72139, 0.26862\rangle$	$\langle 0.60605, 0.38395\rangle$	$\langle 0.25767, 0.73233\rangle$
$\boldsymbol{A_5}$	$\langle 0.83826, 0.15174\rangle$	$\langle 0.337070.65293\rangle$	$\langle 0.14080, 0.84920\rangle$
$\boldsymbol{A_6}$	$\langle 0.48764, 0.50236\rangle$	$\langle 0.87147, 0.11853\rangle$	$\langle 0.49141, 0.49858\rangle$

Table 5.6: IF pairs for 2010

	R	**S**	**B**
$\boldsymbol{A_1}$	$\langle 0.83260, 0.15739\rangle$	$\langle 0.35009, 0.63991\rangle$	$\langle 0.14646, 0.84354\rangle$
$\boldsymbol{A_2}$	$\langle 0.84485, 0.14515\rangle$	$\langle 0.32189, 0.66811\rangle$	$\langle 0.134204, 0.85580\rangle$
$\boldsymbol{A_3}$	$\langle 0.82977, 0.16022\rangle$	$\langle 0.35660, 0.63340\rangle$	$\langle 0.14928, 0.84072\rangle$
$\boldsymbol{A_4}$	$\langle 0.90046, 0.08954\rangle$	$\langle 0.19391, 0.79609\rangle$	$\langle 0.07860, 0.91140\rangle$
$\boldsymbol{A_5}$	$\langle 0.93345, 0.05655, \rangle$	$\langle 0.11798, 0.87201\rangle$	$\langle 0.04560, 0.94439\rangle$
$\boldsymbol{A_6}$	$\langle 0.29820, 0.69180\rangle$	$\langle 0.53591, 0.45410\rangle$	$\langle 0.68086, 0.309146\rangle$

Table 5.7: IF pairs for 2009

	R	**S**	**B**
$\boldsymbol{A_1}$	$\langle 0.55456, 0.43543\rangle$	$\langle 0.99, 0\rangle$	$\langle 0.42449, 0.56550\rangle$
$\boldsymbol{A_2}$	$\langle 0.29442, 0.69557\rangle$	$\langle 0.52923, 0.46076\rangle$	$\langle 0.6846, 0.30537\rangle$
$\boldsymbol{A_3}$	$\langle 0.46313, 0.52686\rangle$	$\langle 0.82806, 0.16193\rangle$	$\langle 0.51591, 0.47408\rangle$
$\boldsymbol{A_4}$	$\langle 0.44334, 0.54665\rangle$	$\langle 0.79300, 0.19699\rangle$	$\langle 0.53571, 0.45428\rangle$
$\boldsymbol{A_5}$	$\langle 0.58095, 0.40904\rangle$	$\langle 0.92926, 0.06073\rangle$	$\langle 0.39810, 0.59189\rangle$
$\boldsymbol{A_6}$	$\langle 0.56775, 0.42224\rangle$	$\langle 0.95963, 0.030368\rangle$	$\langle 0.41130, 0.57869\rangle$

Table 5.8: IF pairs for 2008

	R	**S**	**B**
$\boldsymbol{A_1}$	$\langle 0.77511, 0.21489\rangle$	$\langle 0.48241, 0.50759\rangle$	$\langle 0.20395, 0.78605\rangle$
$\boldsymbol{A_2}$	$\langle 0.84297, 0.14703\rangle$	$\langle 0.32623, 0.66377\rangle$	$\langle 0.13609, 0.85391\rangle$
$\boldsymbol{A_3}$	$\langle 0.19358, 0.79642\rangle$	$\langle 0.35060, 0.63940\rangle$	$\langle 0.78548, 0.20452\rangle$
$\boldsymbol{A_4}$	$\langle 0.80433, 0.18567\rangle$	$\langle 0.41516, 0.57484\rangle$	$\langle 0.17473, 0.81527\rangle$
$\boldsymbol{A_5}$	$\langle 0.80433, 0.18567\rangle$	$\langle 0.41516, 0.57484\rangle$	$\langle 0.17473, 0.81527\rangle$
$\boldsymbol{A_6}$	$\langle 0.50838, 0.48162\rangle$	$\langle 0.90820, 0.08180\rangle$	$\langle 0.47068, 0.51932\rangle$

Table 5.9: IF pairs for 2007

	R	**S**	**B**
$\boldsymbol{A_1}$	$\langle 0.82883, 0.16117\rangle$	$\langle 0.35876, 0.63124\rangle$	$\langle 0.15023, 0.83977\rangle$
$\boldsymbol{A_2}$	$\langle 0.84391, 0.14609\rangle$	$\langle 0.32406, 0.66594\rangle$	$\langle 0.13515, 0.85485\rangle$
$\boldsymbol{A_3}$	$\langle 0.62525, 0.36475\rangle$	$\langle 0.82731, 0.16269\rangle$	$\langle 0.35381, 0.63619\rangle$
$\boldsymbol{A_4}$	$\langle 0.99000, 0.00000\rangle$	$\langle 0.00000, 0.99000\rangle$	$\langle 0.00000, 0.99000\rangle$
$\boldsymbol{A_5}$	$\langle 0.61111, 0.37889\rangle$	$\langle 0.85985, 0.13015\rangle$	$\langle 0.36795, 0.62205\rangle$
$\boldsymbol{A_6}$	$\langle 0.70348, 0.28652\rangle$	$\langle 0.64727, 0.34273\rangle$	$\langle 0.27558, 0.71442\rangle$

Table 5.10: IF pairs for 2006

Step 4:

To aggregate, consider the weighting vector W = (0.1, 0.2, 0.2, 0.2, 0.3). The analysis is focused on making a forecast for the sixth year based on previous five years. The results are shown in Table 5.11.

	R	**S**	**B**
$\boldsymbol{A_1}$	$\langle 0.76474, 0.22450\rangle$	$\langle 0.76293, 0.00043\rangle$	$\langle 0.24366, 0.746198\rangle$
$\boldsymbol{A_2}$	$\langle 0.76865, 0.21966\rangle$	$\langle 0.46445, 0.52369\rangle$	$\langle 0.31518, 0.67381\rangle$
$\boldsymbol{A_3}$	$\langle 0.62472, 0.36353\rangle$	$\langle 0.66926, 0.31845\rangle$	$\langle 0.47021, 0.51827\rangle$
$\boldsymbol{A_4}$	$\langle 0.91056, 0\rangle$	$\langle 0.42796, 0.55990\rangle$	$\langle 0.21181, 0.55990\rangle$
$\boldsymbol{A_5}$	$\langle 0.69315, 0.29082\rangle$	$\langle 0.69440, 0.28823\rangle$	$\langle 0.68860, 0.18853\rangle$
$\boldsymbol{A_6}$	$\langle 0.55609, 0.43340\rangle$	$\langle 0.83318, 0.15077\rangle$	$\langle 0.46524, 0.524287\rangle$

Table 5.11: Forecast for the sixth year using IF_{WMA}

Step 5:

Once matrix is formed, the data is ready for analysis. Assume W = (0.5, 0.2, 0.3). Aggregated result using IF_{WMA} is as follows.

$A_1= \langle 0.665534, 0.09220\rangle$ $A_2=\langle 0.62107, 0.36581\rangle$ $A_3=\langle 0.59421, 0.39377\rangle$

$A_4=\langle 0.75098, 0\rangle$ $A_5=\langle 0.69205, 0.25490\rangle$ $A_6=\langle 0.61404, 0.37152\rangle$

Step 6:

Calculate the score values.

A_1 =0.78666, A_2= 0.62762, A_3 = 0.60021 A_4 =0.87549, A_5=0.71857, A_6 =0.62125

Rank all the alternatives in accordance with the scores:

$$A_4 \succ A_1 \succ A_5 \succ A_2 \succ A_6 \succ A_3$$

Hence, A_4 is the best alternative. It is forecasted that using IF_{WMA}, *Mining & quarrying sector* influence the growth of GDP for the forth coming year 2011.

5.7 Results and discussion

A comparitive analysis has been made using moving average in crisp, fuzzy and intuitionistic fuzzy environment. While observing the given time series data, it is clear that A_4 influences the growth of GDP, which reflects the original forecast for the year 2011. F_{WMA} shows that A_1 is the dominating factor, WMA shows that A_6 is the dominating factor, which are least influencing factor in the original data for 2011.

Hence, it is inferred that, by employing IF_{WMA}, A_4 is the best influencing factor to the growth of GDP and comparitively IF_{WMA} is a better tool to forecast the result for the forth coming year 2011.

Comparison of Ranking orders of alternatives using different moving average operators is as follows

Aggregation operators	**Ranking orders of alternatives**	**Best alternatives**
$\boldsymbol{IF_{WMA}}$	$A_4 \succ A_1 \succ A_5 \succ A_2 \succ A_6 \succ A_3$	A_4
$\boldsymbol{F_{WMA}}$	$A_1 \succ A_6 \succ A_3 \succ A_4 \succ A_2 \succ A_5$	A_1
$\boldsymbol{WMA}$	$A_6 \succ A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$	A_6

In literature, nine types of intutionistic fuzzification functions are defined to characterize fuzziness on the basis of different shapes of the membership and non-membership functions. Here, $iftrif$ and $iftraf$ are used for fuzzification. The choice of intuitionistic fuzzification function to be used depends entirely on the problem under consideration.

Note that in this example, sixth year forecast has been made based on previous five years using IF_{WMA} operator and can also be used in a more dynamic process that considers 1 year, 2 years, 3 years and more. Here, instead of using the moving average for one variable, it is used for the whole matrix.

Chapter 6

Intuitionistic fuzzy tree center-based clustering algorithm

6.1 Introduction

Graph theoretical ideas are highly utilized in data mining, image segmentation, clustering, image processing and networks. Graph theory appears to be very convenient to describe clustering problems. The concept of a tree can be used to design a data structure of a model. The notion of fuzzy sets was introduced by L.A Zadeh as a method of representing uncertainty and vagueness in [110]. The theory of intuitionistic fuzzy sets (IFSs), introduced by Atanassov([8], [9]), is an extension of fuzzy set theory in which, not only membership degree is given, but also non-membership degree, which is more or less independent. Fuzziness and uncertainty in the real world existing information, the attributes of the data sets are often given with intuitionistic fuzzy sets. Intutionistic fuzzy set is a suitable tool to cope with imperfectly defined facts and data, as well as with imprecise knowledge. A.Rosenfeld introduced and examined such concepts as paths, cycle, trees and connectedness in fuzzy graphs [86]. In [66], various types of fuzzy cycles, fuzzy trees in fuzzy graphs defined using level sets. The concept of domination in fuzzy graphs was studied in [95]. R.Parvathi and K.Atanassov [37] defined intuitionistic fuzzy trees using index matrix interpretation. M.Akram and N.O.Alshehri [1] introduced various types of intuitionistic fuzzy trees and investigated some of their properties. Zhang and Chen [114] suggested a clustering technique of IFSs on the basis of the λ-cutting matrix of an interval-valued matrix. Xu and Yager [104] gave a clustering technique by transforming an association matrix into an equivalent association matrix, from which a k-cutting matrix is derived and used to cluster the given IFSs. Cai et al. [33] presented a clustering method based on the intuitionistic fuzzy equivalent dissimilarity matrix and (α, β)-cutting matrices. Zahn [109] proposed clustering algorithm using the minimal spanning tree (MST). Distance between IFSs is considered to form clusters in [109]. Dong et al. [42] gave a hierarchical clustering algorithm based on fuzzy graph connectedness. H.Zhao et al.[115] developed an intuitionistic fuzzy minimum spanning tree clustering algorithm to deal with intuitionistic fuzzy information. Hence, intuitionistic fuzzy clustering techniques are based on distance and similarity measure betweem IFSs. In this way, the authors are motivated to concentrate on *intuitionistic fuzzy trees* (IFTs) and their structure and to apply these concepts to design a clustering algorithm. In this paper, distance, radius, diameter and center of intuitionistic fuzzy trees are introduced and their domination properties are analyzed. Also, intuitionistic fuzzy tree center-based clustering algorithm is proposed to cluster the numerical data set. As the existing data in real-life are crisp, S-shaped intuitionistic fuzzification function is used in the proposed method. These values give the membership and non-membership of the vertices of the IFT under consideration. A new distance measure [1] between two IFSs, is defined and applied it to construct the intuitionistic fuzzy distance matrix. Center of an IFT is obtained by eccentricity concept. On the basis of the (λ, δ)-cutting matrix on distance matrix is used to cluster the given dataset. This algorithm is verified with classification of the numerical data sets containing nutrients in 27 different kinds of meat, fish or fowl with five attributes. The developed clustering method is compared with two existing clustering methods namely Zhang et al. [114] and Z.Wang et al. [116].

6.2 Preliminaries

In this section, some basic definitions relating to intuitionistic fuzzy graphs (IFGs) are given. Also, the definitions of partial spanning subgraph, spanning subgraph, distance, eccentricity, radius, diameter and center of IFTs are given.

Notations

1. Hereafter, (μ_i, ν_i) denotes the degrees of membership and non-membership of the vertex $v_i \in V$ such that $0 \leq \mu_i + \nu_i \leq 1$.

2. (μ_{ij}, γ_{ij}) denotes the degrees of membership and non-membership of the edge $(v_i, v_j) \in V \times V$ such that $0 \leq \mu_{ij} + \gamma_{ij} \leq 1$.

Definition 6.2.1. [95] Let X be a universal set and let V be an IFS over X in the form $V = \{(v_i, \mu_i, \gamma_i) \,|v_i \in V\}$ such that $0 \leq \mu_i + \gamma_i \leq 1$. *Six types of cartesian products* of n elements of V over X are defined as

[1]The proposed measure is characterized by interval of membership and non-membership values rather a classical distance measure, which is defined as real number.

$$v_1 \times_1 v_2 \times_1 v_3 \times_1 \cdots \times_1 v_n = \left\{ \left\langle (v_1, v_2, \cdots, v_n), \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i \right\rangle | (v_1, v_2, \cdots, v_n) \in V \right\}$$

$$v_1 \times_2 v_2 \times_2 v_3 \times_2 \cdots \times_2 v_n = \left\{ \left\langle (v_1, v_2, \ldots, v_n), \sum_{i=1}^{n} \mu_i - \sum_{i \neq j}^{n} \mu_i \mu_j + \sum_{i \neq j \neq k}^{n} \mu_i \mu_j \mu_k - \right.\right.$$

$$\left.\left. \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \neq n}^{n} \mu_i \mu_j \mu_k \cdots \mu_n + (-1)^{n-1} \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i \right\rangle | (v_1, v_2, \cdots, v_n) \in V \right\}$$

$$v_1 \times_3 v_2 \times_3 v_3 \times_3 \cdots \times_3 v_n = \left\{ \left\langle (v_1, v_2, \cdots, v_n), \prod_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \gamma_i - \sum_{i \neq j}^{n} \gamma_i \gamma_j + \right.\right.$$

$$\left.\left. \sum_{i \neq j \neq k}^{n} \gamma_i \gamma_j \gamma_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \neq n}^{n} \gamma_i \gamma_j \gamma_k \cdots \gamma_n + (-1)^{n-1} \prod_{i=1}^{n} \gamma_i \right\rangle | (v_1, v_2, \cdots, v_n) \in V \right\}$$

$$v_1 \times_4 v_2 \times_4 v_3 \times_4 \cdots \times_4 v_n =$$
$$\{\langle (v_1, v_2, \cdots, v_n), min(\mu_1, \mu_2, \cdots, \mu_n), \quad max(\gamma_1, \gamma_2, \cdots, \gamma_n) \rangle | (v_1, v_2, \cdots, v_n) \in V\}$$

$$v_1 \times_5 v_2 \times_5 v_3 \times_5 \cdots \times_5 v_n =$$
$$\{\langle (v_1, v_2, \cdots, v_n), max(\mu_1, \mu_2, \cdots, \mu_n), \quad min(\gamma_1, \gamma_2, \cdots, \gamma_n) \rangle | (v_1, v_2, \cdots, v_n) \in V\}$$

$$v_1 \times_6 v_2 \times_6 v_3 \times_6 \cdots \times_6 v_n = \left\{ \left\langle \langle v_1, v_2, \ldots, v_n \rangle, \frac{\sum_{i=1}^{n} \mu_i}{n}, \frac{\sum_{i=1}^{n} \gamma_i}{n} \right\rangle | \langle v_1, v_2, \cdots, v_n \rangle \in V \right\}$$

It must be noted that $v_i \times_t v_j$ is an IFS , where $t = 1, 2, 3, 4, 5, 6$ such that the sum of their degrees of membership and non-membership lies in $[0, 1]$.

Definition 6.2.2. [53] An *intuitionistic fuzzy graph* (IFG) is of the form $G = (V, E)$ where

(i) $V = \{v_1, v_2, ..., v_n\}$, such that $\mu_i : V \to [0, 1]$ and $\gamma_i : V \to [0, 1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively, and $0 \leq \mu_i + \gamma_i \leq 1$ for every $v_i \in V$, $i = 1, 2, \cdots, n$

(ii) $E \subset V \times V$ where $\mu_{ij} : V \times V \to [0,1]$ and $\gamma_{ij} : V \times V \to [0,1]$ are such that

$$\mu_{ij} \leq \mu_i \oslash \mu_j,$$

$$\gamma_{ij} \leq \gamma_i \oslash \gamma_j$$

and

$$0 \leq \mu_{ij} + \gamma_{ij} \leq 1$$

where μ_{ij} and γ_{ij} are the degrees of membership and non-membership of the edge (v_i, v_j); the values $\mu_i \oslash \mu_j$ and $\gamma_i \oslash \gamma_j$ can be determined by one of the six cartesian products $\times_t$, $\quad t = 1, 2, 3, 4, 5, 6$ for all i and j given in Definition 2.

Definition 6.2.3. An IFG, $H = (V', E')$ is said to be a *partial intuitionistic fuzzy subgraph* of $G = (V, E)$ if

(i) $V' \subset V$, $\mu_i^{'} \leq \mu_i$, $\gamma_i^{'} \leq \gamma_i$ for all $v_i \in V^{'}, i = 1, 2, \ldots n$.

(ii) $E' \subset E$, $\mu_{ij}^{'} \leq \mu_{ij}$, $\gamma_{ij}^{'} \leq \gamma_{ij}$ for all $(v_i, v_i) \in E^{'}, i, j = 1, 2, \ldots n$.

Definition 6.2.4. [53] An IFG, $H = (V', E')$ is said to be an *intuitionistic fuzzy subgraph* of $G = (V, E)$ if

(i) $V' \subset V$, $\mu_i^{'} = \mu_i$, $\gamma_i^{'} = \gamma_i$ for all $v_i \in V^{'}, i = 1, 2, \ldots n$.

(ii) $E' \subset E$, $\mu_{ij}^{'} = \mu_{ij}$, $\gamma_{ij}^{'} = \gamma_{ij}$ for all $(v_i, v_i) \in E^{'}, i, j = 1, 2, \ldots n$.

Definition 6.2.5. An IFG, $H = (V', E')$ is said to be a *partial intuitionistic fuzzy spanning subgraph* of $G = (V, E)$ if

(i) $V' = V$, $\mu_i^{'} \leq \mu_i$, $\gamma_i^{'} \leq \gamma_i$ for all $v_i \in V^{'}, i = 1, 2, \ldots n$.

(ii) $E' \subset E$, $\mu_{ij}^{'} \leq \mu_{ij}$, $\gamma_{ij}^{'} \leq \gamma_{ij}$ for all $(v_i, v_i) \in E^{'}, i, j = 1, 2, \ldots n$.

Definition 6.2.6. An IFG, $H = (V', E')$ is said to be an *intuitionistic fuzzy spanning subgraph*(IFSSG) of $G = (V, E)$ if

(i) $V' = V$, $\mu_i^{'} = \mu_i$, $\gamma_i^{'} = \gamma_i$ for all $v_i \in V^{'}, i = 1, 2, \ldots n$.

(ii) $E' \subset E$, $\mu_{ij}^{'} = \mu_{ij}$, $\gamma_{ij}^{'} = \gamma_{ij}$ for all $(v_i, v_i) \in E^{'}, i, j = 1, 2, \ldots n$.

Definition 6.2.7. [77] Let $G = (V, E)$ be an IFG, then the *cardinality* of a subset S of V is defined as $|S| = \sum_{v_i \in S} \left(\frac{1 + \mu_i - \gamma_i}{2} \right)$ for all $v_i \in S$.

Definition 6.2.8. [77] The number of vertices in G is called as *order* of an IFG, $G = (V, E)$, denoted by $o(G)$, and is defined as $o(G) = \sum_{v_i \in V} \left(\frac{1 + \mu_i - \gamma_i}{2} \right)$ for all $v_i \in V$.

Definition 6.2.9. [77] An IFG, $G = (V, E)$ is said to be *complete IFG* if $\mu_{ij} = \min(\mu_i, \mu_j)$ and $\gamma_{ij} = \max(\gamma_i, \gamma_j)$ for every $v_i, v_j \in V$.

Definition 6.2.10. [66] If $v_i, v_j \in V \subseteq G$, the μ-strength of connectedness between v_i and v_j is $\mu_{ij}^{\infty} = \sup\{\mu_{ij}^k \mid k = 1, 2, \ldots, n\}$ and γ-strength of connectedness between v_i and v_j is $\gamma_{ij}^{\infty} = \inf\{\gamma_{ij}^k \mid k = 1, 2, \ldots, n\}$.
If v_i, v_j are connected by means of paths of length k then μ_{ij}^k is defined as $\sup\{\mu_{i1} \wedge \mu_{12} \wedge \mu_{23} \ldots \wedge \mu_{k-1j} \mid v_i, v_1, v_2 \ldots v_{k-1}, v_j \in V\}$ and γ_{ij}^k is defined as $\inf\{\gamma_{i1} \vee \gamma_{12} \vee \gamma_{23} \ldots \vee \gamma_{k-1j} \mid v_i, v_1, v_2 \ldots v_{k-1}, v_j \in V\}$

Definition 6.2.11. [66] An edge (v_i, v_j) is said to be a *strong edge* of an IFG $G = (V, E)$, if $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$.

Definition 6.2.12. [95] An IFG, $G = (V, E)$ is said to be *connected* IFG if there exists a path between every pair of vertices v_i, v_j in V. Connected IFG is also defined using strength of connectedness as follows:

(i) $\mu_{ij}^{\infty} > 0$, and $\gamma_{ij}^{\infty} > 0$

(ii) $\mu_{ij}^{\infty} = 0$, and $\gamma_{ij}^{\infty} > 0$

(iii) $\mu_{ij}^{\infty} > 0$, and $\gamma_{ij}^{\infty} = 0$ for all $v_i, v_j \in V$.

Definition 6.2.13. [99] An IFG, $G = (V, E)$ is said to be *intuitionistic fuzzy forest* (IFF), if it has an intuitionistic fuzzy spanning subgraph $H = (V', E')$, which is a forest (in crisp sense), where for all edges (v_i, v_j) not in H, $\mu_{ij} < \mu_{ij}^{'\infty}$ and $\gamma_{ij} > \gamma_{ij}^{'\infty}$.

Definition 6.2.14. [99] An connected IFG, $G = (V, E)$ is said to be *intuitionistic fuzzy tree* (IFT) if it has an intuitionistic fuzzy spanning subgraph $H = (V', E')$, which is a tree (in crisp sense), where for all edges (v_i, v_j) not in H, $\mu_{ij} < \mu_{ij}^{'\infty}$ and $\gamma_{ij} > \gamma_{ij}^{'\infty}$.

Definition 6.2.15. [99] A connected IFG, $G = (V, E)$ is said to be *intuitionistic fuzzy spanning tree* (IFST), if it has an IFSSG, $H = (V', E')$ which is a tree.

Definition 6.2.16. [77] A *path* in an IFG is a sequence of distinct vertices $v_1, v_2, \ldots v_n$, such that either one of the following conditions is satisfied for some $i, j = 1, 2, 3 \ldots n$:

(i) $\mu_{ij} > 0, \gamma_{ij} > 0$

(ii) $\mu_{ij} = 0, \gamma_{ij} > 0$

(iii) $\mu_{ij} > 0, \gamma_{ij} = 0.$

Definition 6.2.17. [99] A *strong path* in an IFG is a path $P = v_1v_2\ldots v_n$, in which for every edge $(v_i, v_j) \in P$, is strong edge.

Definition 6.2.18. [42] The *length* of a path $P = v_1v_2\ldots v_{n+1}$ $(n > 0)$ is n.

Example 6.2.1. Consider an IFG, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_4), (v_1, v_3), (v_3, v_4)\}$. and its IFSSG $H = V', E')$, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_4), (v_1, v_3)\}$. This example shows an IFG is an IFT

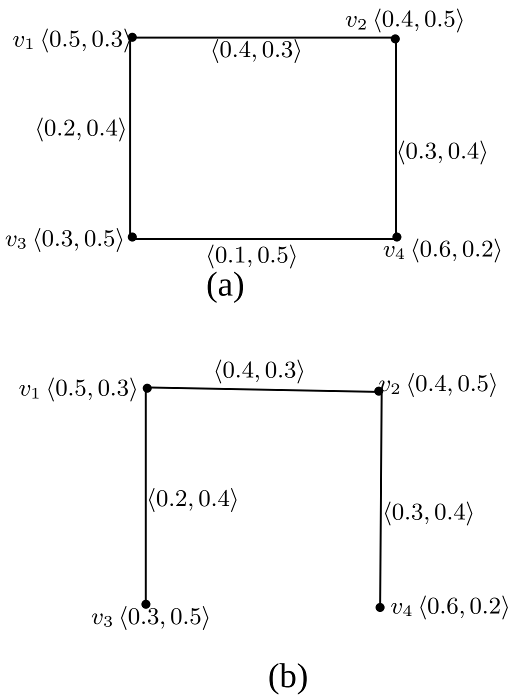


Figure 6.1: (a) Intuitionistic fuzzy graph G (b)Spanning subgraph H

Note 1. Every Intuitionistic fuzzy graph is not an intuitionistic fuzzy tree.

Example 6.2.2. Consider an IFG, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_1, v_4), (v_2, v_3)\}$.

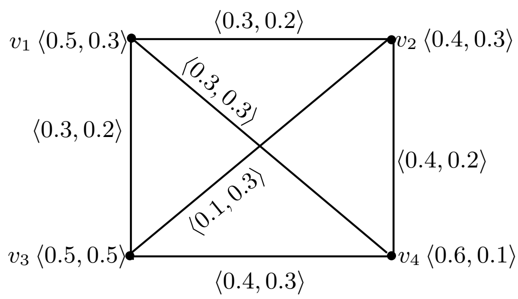


Figure 6.2: Intuitionistic fuzzy graph G

In the above example, Figure 6.2 is an intuitionistic fuzzy graph but not an intuitionistic fuzzy tree.

Definition 6.2.19. Let $G = (V, E)$ be an IFT and let $P = v_1 v_2 \ldots v_n$ be a path. The *μ-length* of P in G, denoted by $l_\mu(P)$, is defined as, $l_\mu(P) = \sum_{(v_i, v_j) \in P} \mu_{ij}$, $i, j = 1, 2, 3 \ldots n$. The *γ- length* of path P, in G, denoted by $l_\gamma(P)$, is defined as $l_\gamma(P) = \sum_{(v_i, v_j) \in P} \gamma_{ij}$, $i, j = 1, 2, 3 \ldots n$. The *length* of P in G, denoted by $l(P)$, is defined as $l(P) = (l_\mu(P), l_\gamma(P))$.

Definition 6.2.20. Let $G = (V, E)$ be an IFT. For any two vertices v_i and v_j in G, Let $\Omega = \{P_i : P_i \text{ is a } v_i - v_j \text{path}, i = 1, 2, 3 \ldots n\}$. The *$\mu$- distance* between any two vertices $v_i, v_j \in V$, denoted by δ_{μ_i, μ_j} and is defined as

$\delta_{\mu_i,\mu_j} = \min \{l_\mu(P_i) : P_i \in \Omega, i = 1, 2, 3 \ldots n\}$.

The γ- *distance* between any two vertices $v_i, v_j \in V$, denoted by $\delta_{\gamma_i,\gamma_j}$ and is defined as $\delta_{\gamma_i,\gamma_j} = \min \{l_\gamma(P_i) : P_i \in \Omega, i = 1, 2, 3 \ldots n\}$. The *distance*, $\delta(v_i, v_j)$, is defined as $\delta(v_i, v_j) = (\delta_{\mu_i,\mu_j}, \delta_{\gamma_i,\gamma_j})$.

Definition 6.2.21. Let $G = (V, E)$ be an IFT. For each $v_i \in V$, the μ- *eccentricity* of v_i, denoted by e_{μ_i}, is defined as $e_{\mu_i}= \max\left\{\delta_{\mu_i,\mu_j} : v_i \in V, v_i \neq v_j\right\}$. For each $v_i \in V$, the γ-*eccentricity* of v_i, denoted by e_{γ_i}, is defined as $e_{\gamma_i}= \max \left\{\delta_{\gamma_i,\gamma_j} : v_i \in V, v_i \neq v_j\right\}$. For each $v_i \in V$, the *eccentricity* of v_i, denoted by $e(v_i)$, and is defined as $e(v_i) = (e_{\mu_i}, e_{\gamma_i})$.

Definition 6.2.22. Let $G = (V, E)$ be an IFT. The μ-*radius* of G, denoted by $r_\mu(G)$, is defined as $r_\mu(G) = \min \{e_{\mu_i} : v_i \in V\}$. The γ - *radius* of G, denoted by $r_\gamma(G)$, is defined as $r_\gamma(G) = \min \{e_{\gamma_i} : v_i \in V\}$.The *radius* of G, denoted by $r(G)$ is defined as $r(G) = (r_\mu(G), r_\gamma(G))$.

Definition 6.2.23. Let $G = (V, E)$ be an IFT. The μ-*diameter* of G, denoted by $diam_\mu(G)$, is defined as $diam_\mu(G) = \max \{e_{\mu_i} : v_i \in V\}$. The γ-*diameter* of G, denoted by $diam_\gamma(G)$, defined as $diam_\gamma(G) = \max \{e_{\gamma_i} : v_i \in V\}$. The *diameter* of G, denoted by diam(G), is defined as diam $(G) = (diam_\mu(G), diam_\gamma(G))$.

Definition 6.2.24. A vertex $v_i \in V$ is called a *central* vertex of an IFT $G = (V, E)$, if $r_\mu(G) = e_{\mu_i}$ and $r_\gamma(G) = e_{\gamma_i}$. The set of all central vertices of an IFT is denoted by $C_V(G)$

Definition 6.2.25. An IFSG $H = (V', E')$ induced by the central vertices of G , is called *center* of G, denoted by $C(G)$.

Definition 6.2.26. A vertex $v_i \in V$ is called a *peripheral* vertex of an IFT $G = (V, E)$, if $diam_\mu(G) = e_{\mu_i}$ and $diam_\gamma(G) = e_{\gamma_i}$. The set of all peripheral vertices of an IFT is denoted by $Z(G)$.

Definition 6.2.27. Let $G = (V, E)$ be an IFT, then the distance function $\delta :$ $V \times V \to [0, 1] \times [0, 1]$ is a *metric* on V , if the following conditions are satisfied:

(i) $\delta(v_i, v_j) \geq 0$

ie, $\delta_{\mu_i,\mu_j} \geq 0$, $\delta_{\gamma_i,\gamma_j} \geq 0$, $\forall v_i, v_j \in V$

(ii) $\delta(v_i, v_j) = (0, 1)$ if and only if $v_i = v_j$

(iii) $\delta(v_i, v_j) = \delta(v_j, v_i)$ ie, $\delta_{\mu_i,\mu_j} = \delta_{\mu_j,\mu_i}$, $\delta_{\gamma_i,\gamma_j} = \delta_{\gamma_j,\gamma_i}$

(iv) $\delta_{\mu_i,\mu_j} \leq \delta_{\mu_j,\mu_k} + \delta_{\mu_k,\mu_j}$, $\delta_{\gamma_i,\gamma_j} \leq \delta_{\gamma_j,\gamma_k} + \delta_{\gamma_k,\gamma_j}$, $v_i, v_j, v_k \in V$.

Definition 6.2.28. [3] A vertex $v_k \in V$ of an IFG $G = (V, E)$ is called *cut vertex* if $\mu^\infty_{ij}(G - v_k) < \mu^\infty_{ij}$ and $\gamma^\infty_{ij}(G - v_k) > \gamma^\infty_{ij}$ for some $v_i, v_j \in V$.

Definition 6.2.29. Let $G = (V, E)$ be an IFG and let $Y = \{v_1, v_2, \ldots v_n\}$ be vertex cut in G. The μ- *strong weight* of Y in G, denoted by $S_\mu(Y)$, is defined as, $S_\mu(Y) = \sum_{v_j \in Y} \mu_{ij}$, $i, j = 1, 2, 3 \ldots n$, where μ_{ij} is the minimum membership weight of strong edges incident on v_i.The γ- *strong weight* of Y in G, denoted by $S_\gamma(Y)$, is defined as, $S_\mu(Y) = \sum_{v_j \in Y} \mu_{ij}$, $i, j = 1, 2, 3 \ldots n$, where γ_{ij} is the maximum non-membership weight of strong edges incident on v_i. The *strong weight* of Y in G, denoted by $S(Y)$, is defined as $S(Y) = (S_\mu(Y), S_\gamma(Y))$.

Definition 6.2.30. Let $G = (V, E)$ be an IFG, the μ-*vertex connectivity* of G, denoted by $k_\mu(G)$, is defined as, $k_\mu(G) = \min(S_\mu(Y))$. The γ-*vertex connectivity* of

G, denoted by $k_\gamma(G)$, is defined as, $k_\gamma(G) = \min(S_\gamma(Y))$. The *vertex connectivity* of G, denoted by $k(G)$, is defined as, $(k_\mu(G), k_\gamma(G))$.

Definition 6.2.31. An edge $e_k \in E$ of an IFG $G = (V, E)$ is called *cut edges* if $\mu_{ij}^\infty(G - e_k) < \mu_{ij}^\infty$ and $\gamma_{ij}^\infty(G - e_k) > \gamma_{ij}^\infty$ for some $v_i, v_j \in V$.

Definition 6.2.32. Let $G = (V, E)$ be an IFG and let $E = \{e_1, e_2, \ldots e_n\}$ be edge cut in G. The μ- *strong weight* of E in G, denoted by $S_\mu^{'}(E)$, is defined as, $S_\mu^{'}(E) = \sum_{e_i \in E} \mu_{ij}$, $i, j = 1, 2, 3 \ldots n$. The γ- *strong weight* of E in G, denoted by $S_\gamma^{'}(E)$, is defined as, $S_\gamma^{'}(E) = \sum_{e_i \in E} \gamma_{ij}$, $i, j = 1, 2, 3 \ldots n$, The *strong weight* of E in G, denoted by $S^{'}(E)$, is defined as $S^{'}(E) = (S_\mu^{'}(E), S_\gamma^{'}(E))$.

Definition 6.2.33. Let $G = (V, E)$ be an IFG, the μ-*edge connectivity* of G, denoted by $k_\mu^{'}(G)$, is defined as, $k_\mu^{'}(G) = \min(S_\mu^{'}(E))$. The γ-*edge connectivity* of G, denoted by $k_\gamma^{'}(G)$, is defined as, $k_\gamma^{'}(G) = \min(S_\gamma^{'}(E))$. The *edge connectivity* of G, denoted by $k^{'}(G)$, is defined as, $(k_\mu^{'}(G), k_\gamma^{'}(G))$.

Definition 6.2.34. A vertex $v_i \in V$ of an IFT $G = (V, E)$ is called *end vertex* if $\mu_{ij} \geq \mu_{ij}^\infty$ and $\gamma_{ij} \geq \gamma_{ij}^\infty$ for at most one $v_j \in V$.

Note 2. (1) In an IFT, the diameter not necessarily be twice of the radius.

(2) The center of intuitionistic fuzzy tree need not be k_1 or k_2.

(3) For any spanning subgraph H (which is a tree) of G contains atleast two end vertices and every vertex in G is either cut vertex or end vertex.

Example 6.2.3. Example for single-centered intuitionistic fuzzy tree with one central vertices

Consider an IFT, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_1, v_5), (v_3, v_4), (v_5, v_4), (v_3, v_2)\}$.

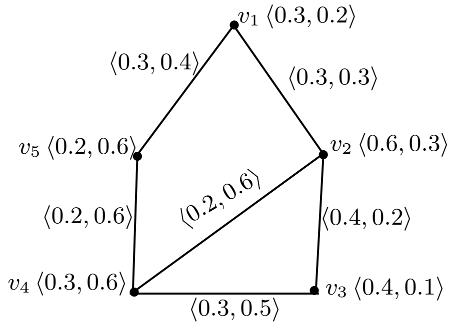


Figure 6.3: Intuitionistic fuzzy tree G

By routine computations , we have

(i) $\delta(v_1, v_2) = \langle 0.3, 0.3\rangle$, $\delta(v_1, v_3) = \langle 0.7, 0.5\rangle$, $\delta(v_1, v_4) = \langle 0.5, 0.9\rangle$, $\delta(v_1, v_5) = \langle 0.3, 0.4\rangle$, $\delta(v_2, v_3) = \langle 0.4, 0.2\rangle$, $\delta(v_2, v_4) = \langle 0.2, 0.6\rangle$, $\delta(v_2, v_5) = \langle 0.4, 0.6\rangle$, $\delta(v_3, v_4) = (0.3, 0.5)$, $\delta(v_3, v_5) = \langle 0.5, 0.9\rangle$, $\delta(v_4, v_5) = \langle 0.2, 0.6\rangle$.

(ii) Eccentricity of each vertex is $e(v_1) = \langle 0.7, 0.9\rangle$, $e(v_2) = \langle 0.4, 0.6\rangle$, $e(v_3) = \langle 0.7, 0.9\rangle$, $e(v_4) = \langle 0.5, 0.9\rangle$, $e(v_5) = \langle 0.5, 0.9\rangle$

(iii) Radius of G is $\langle 0.4, 0.6\rangle$, diameter of G is $\langle 0.7, 0.9\rangle$.

(iv) The central vertex of G is v_2, That is $r(G) = e(v_2)$

(v) The center of G is displayed in Figure 6.4

(vi) The peripheral vertices of G are v_3 and v_1.

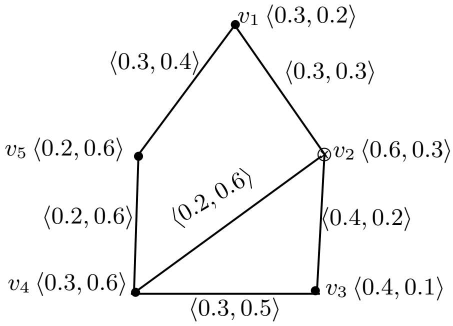


Figure 6.4: Center $C(G)$

Example 6.2.4. Example for bi-centered intuitionistic fuzzy tree with two central vertices:

Consider an IFT, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$,

$E = \{(v_1, v_2), (v_4, v_3), (v_2, v_4), (v_1, v_4), (v_2, v_3)\}$.

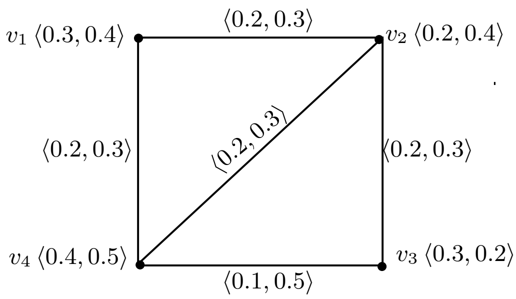


Figure 6.5: Intuitionistic fuzzy tree G

Therefore,

(i) $\delta(v_1, v_2) = \langle 0.2, 0.3 \rangle$, $\delta(v_1, v_3) = \langle 0.3, 0.6 \rangle$, $\delta(v_1, v_4) = \langle 0.2, 0.3 \rangle$, $\delta(v_2, v_1) = \langle 0.2, 0.3 \rangle$, $\delta(v_2, v_4) = \langle 0.1, 0.5 \rangle$, $\delta(v_2, v_3) = \langle 0.2, 0.3 \rangle$, $\delta(v_3, v_4) = \langle 0.1, 0.5 \rangle$

(ii) Eccentricity of each vertex is $e(v_1) = \langle 0.3, 0.6 \rangle$, $e(v_2) = \langle 0.2, 0.5 \rangle$, $e(v_3) = \langle 0.3, 0.6 \rangle$, $e(v_4) = \langle 0.2, 0.5 \rangle$

(iii) Radius of G is $\langle 0.2, 0.5\rangle$, diameter of G is $\langle 0.3, 0.6\rangle$.

(iv) The central vertices of G is $\{v_2, v_4\}$, That is $r(G) = e(v_2)$, $r(G) = e(v_4)$

(v) The center of $C(G)$ is displayed in Figure 6.6.

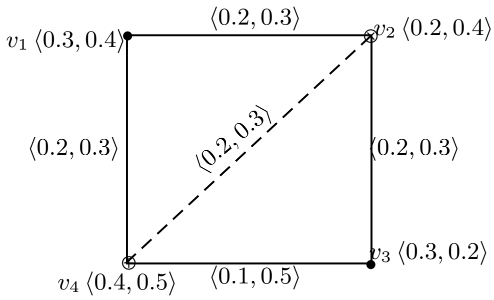


Figure 6.6: Center $C(G)$

(vi) The peripheral vertices of G are v_1 and v_3.

Example 6.2.5. Example for tri-centered intuitionistic fuzzy tree with three central vertices:

Consider an IFT, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4, v_5\}$,

$E = \{(v_1, v_2), (v_4, v_3), (v_2, v_4), (v_1, v_4), (v_2, v_3), (v_5, v_2), (v_1, v_5), (v_5, v_4), (v_1, v_5)\}$.

By routine computations , we have

(i) $\delta(v_1, v_2) = \langle 0.5, 0.5\rangle$, $\delta(v_1, v_3) = \langle 0.5, 0.2\rangle$, $\delta(v_1, v_4) = \langle 0.3, 0.4\rangle$, $\delta(v_1, v_5) = \langle 0.5, 0.3\rangle$ $\delta(v_2, v_3) = \langle 0.2, 0.5\rangle$, $\delta(v_2, v_4) = \langle 0.3, 0.4\rangle$, $\delta(v_2, v_5) = \langle 0.3, 0.4\rangle$, $\delta(v_3, v_4) = \langle 0.2, 0.5\rangle$, $\delta(v_3, v_5) = \langle 0.5, 0.7\rangle$, $\delta(v_4, v_5) = \langle 0.5, 0.5\rangle$

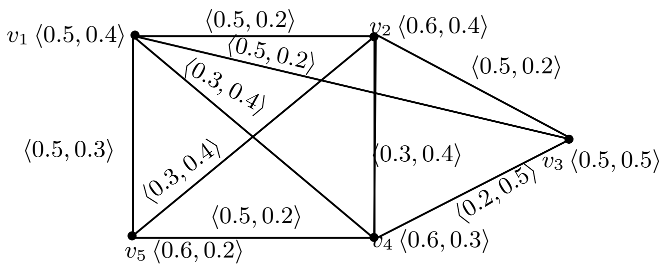


Figure 6.7: Intuitionistic fuzzy tree G

(ii) Eccentricity of each vertex is $e(v_1) = \langle 0.5, 0.5\rangle$, $e(v_2) = \langle 0.5, 0.5\rangle$, $e(v_3) = \langle 0.5, 0.7\rangle$, $e(v_4) = \langle 0.5, 0.5\rangle$, $e(v_5) = \langle 0.5, 0.7\rangle$

(iii) Radius of G is $\langle 0.5, 0.5\rangle$, diameter of G is $\langle 0.5, 0.7\rangle$.

(iv) The central vertices $C(G)$ is $\{v_1, v_2, v_4\}$, That is $r(G) = e(v_1)$, $r(G) = e(v_2)$, $r(G) = e(v_4)$

(v) The center of $C(G)$ is displayed in Figure 6.8

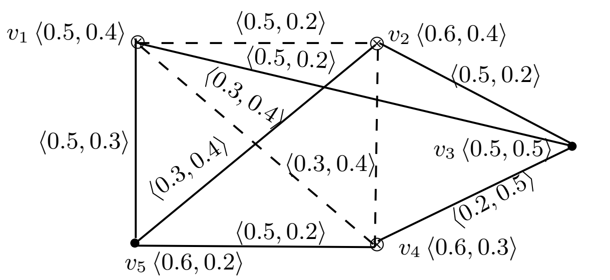


Figure 6.8: Center $C(G)$

(vi) The peripheral vertices of G are v_3 and v_5.

6.3 Domination in intuitionistic fuzzy trees

Definition 6.3.1. [77] Let $G = (V, E)$ be an IFG on V. Let $u, v \in V$, u is said to *dominate* v in G if there exists a strong edge between them.

Definition 6.3.2. [77] A subset S of V is called a *dominating set* in G if for every $v \in V - S$, there exists $u \in S$ such that u dominates v.

Definition 6.3.3. [77] A dominating set S of an IFG is said to be a *minimal dominating set* if no proper subset of S is a dominating set.

Definition 6.3.4. [77] Minimum cardinality among all minimal dominating set is called *lower domination number* of G, and is denoted by $d(G)$.
Maximum cardinality among all minimal dominating set is called *upper domination number* of G, and is denoted by $D(G)$.

Definition 6.3.5. [77] Two vertices in an IFG, $G = (V, E)$ are said to be *independent* if there is no strong edge between them.

Definition 6.3.6. [77] A subset S of V is said to be *independent set* of G if $\mu_{ij} < \mu_{ij}^{\infty}$ and $\gamma_{ij} < \gamma_{ij}^{\infty}$ for all $v_i, v_j \in S$. An independent set S of G in an IFG is said to be *maximal independent*, if for every vertex $v_j \in V - S$, the set $S \cup \{v_j\}$ is not independent.

Definition 6.3.7. [77] The minimum cardinality among all maximal independent set is called *lower independence number* of G, and it is denoted by $i(G)$.
The maximum cardinality among all maximal independent set is called *upper independence number* of G, and it is denoted by $I(G)$.

Definition 6.3.8. [77] Let $G = (V, E)$ be an IFG without isolated vertices. A subset D of V is a *total dominating set* if for every vertex $v_i \in V$, there exists a vertex $v_j \in D$, $v_i \neq v_j$, such that v_j dominates v_i.

Definition 6.3.9. [77] The minimum cardinality of a total dominating set is called *total domination number* of G, and it is denoted by $d_t(G)$.

Definition 6.3.10. [99] Let G be a connected IFG. A subset V' of V is called a *connected dominating set* of G, if

(i) For every $v_j \in V$- V', there exists $v_i \in V'$ such that $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$

(ii) The sub graph $H = (V', E')$ of G=(V, E) induced by V' is connected.

Definition 6.3.11. [99] The minimum cardinality of a connected dominating set is called the *connected domination number* of G, and is denoted by $d_c(G)$.

Example 6.3.1. Consider an IFT, $G = (V, E)$, in Figure 6.3 in Example 6.2.3

(i) The minimal dominating set of G is $\{v_1, v_4\}$ and the domination number $d(G)$ is 0.9.

(ii) The maximal independent set of G is $\{v_1, v_4\}$ and the independent domination number $i(G)$ is 0.9.

(iii) The total dominating set of G is $\{v_1, v_4\}$ and the total domination number $d_t(G)$ is 0.9.

(iv) The connected dominating set of G is $\{v_1, v_2, v_3\}$ and the connected domination number $d_c(G)$ is 0.85

Theorem 6.3.12. Let $G = (V, E)$ is an IFT, then the distance between any two vertices in V is metric.

Proof:

Let $G = (V, E)$ is an IFT, it is connected. Then there exists a unique strong path between any two vertices in V.

That is, $\delta_{\mu_i,\mu_j} \geq 0$, $\delta_{\gamma_i,\gamma_j} \geq 0$, which implies $\delta(v_i, v_j) \geq 0$ for every $v_i, v_j \in V$.

$\delta_{\mu_i,\mu_i} = 0$, $\delta_{\gamma_i,\gamma_i} = 0$, this implies that $\delta(v_i, v_j) = 0$.

Next the reversal of a path from v_iv_j is a path from v_jv_i and vice versa. That is, $\delta_{\mu_i,\mu_i} = 0$, $\delta_{\gamma_i,\gamma_i} = 0$, implies that $\delta(v_i, v_j) = \delta(v_j, v_i)$. Let P_1 is a $v_i - v_k$ path in G such that $\delta_{\mu_i,\mu_k} = \sum\limits_{(v_i,v_k)\in P_1} \mu_{ik}$, $\delta_{\gamma_i,\gamma_k} = \sum\limits_{(v_i,v_k)\in P_1} \gamma_{ik}$,and P_2 be a v_kv_j path such that $\delta_{\mu_k,\mu_j} = \sum\limits_{(v_k,v_j)\in P_2} \gamma_{kj}$, $\delta_{\gamma_k,\gamma_j} = \sum\limits_{(v_k,v_j)\in P_2} \gamma_{kj}$. The path P_1 followed by P_2 is a v_iv_j walk and since every walk contains one path, there exists a v_iv_j path in G whose length is at most $\delta_{\mu_i,\mu_k} + \delta_{\mu_k,\mu_j}, \delta_{\gamma_i,\gamma_k} + \delta_{\gamma_k,\gamma_j}$. Therefore, $\delta_{\mu_i,\mu_j} \leq \delta_{\mu_j,\mu_k} + \delta_{\mu_k,\mu_j}$, $\delta_{\gamma_i,\gamma_j} \leq \delta_{\gamma_j,\gamma_k} + \delta_{\gamma_k,\gamma_j}$. This implies that $\delta(v_i, v_j) \leq \delta(v_i, v_k) + \delta(v_k, v_j)$. Hence the distance δ is a metric on V.

Theorem 6.3.13. For any IFT $G = (V, E)$, the radius and diameter satisfy $r_\mu(G) \leq diam_\mu(G) \leq 2r_\mu(G)$ and $r_\gamma(G) \leq diam_\gamma(G) \leq 2r_\gamma(G)$.

Proof:

By the definition of radius and diameter $r_\mu(G) \leq diam_\mu(G)$ and $r_\gamma(G) \leq diam_\gamma(G)$.

Let v_k be a central vertex and v_i, v_j be two peripheral vertices of G. Then $r_\mu(G) = e_\mu(v_k)$, $r_\gamma(G) = e_\gamma(v_k$ and $diam_\mu(G) = e_\mu(v_i)$,$diam_\gamma(G) = e_\gamma(v_i)$ $diam_\mu(G) = e_\mu(v_j)$, $diam_\gamma(G) = e_\gamma(v_j)$.

By triangle inequality,

$$\delta_{\mu_i,\mu_j} \leq \delta_{\mu_j,\mu_k} + \delta_{\mu_k,\mu_j}$$
$$= r_\mu(G) + r_\mu(G) = 2r_\mu(G)$$
$$\delta_{\gamma_i,\gamma_j} \leq \delta_{\gamma_j,\gamma_k} + \delta_{\gamma_k,\gamma_j},$$
$$= r_\gamma(G) + r_\gamma(G) = 2r_\gamma(G)$$

Therefore, $r_\mu(G) \leq diam_\mu(G) \leq 2r_\mu(G)$ and $r_\gamma(G) \leq diam_\gamma(G) \leq 2r_\gamma(G)$.

Theorem 6.3.14. For any two vertices v_i, v_j in an IFT $G = (V, E)$, $|e_\mu(v_i) - e_\mu(v_j)| \leq \delta_{\mu_i,\mu_j}$ and $|e_\gamma(v_i) - e_\gamma(v_j)| \leq \delta_{\gamma_i,\gamma_j}$.

Proof:

By the definition, of the eccentricity of a vertex v_i in an IFT,

$e_{\mu_i} = \max\left\{\delta_{\mu_i,\mu_j} : v_i \in V, v_i \neq v_j\right\}$ and $e_{\gamma_i} = \max\left\{\delta_{\gamma_i,\gamma_j} : v_i \in V, v_i \neq v_j\right\}$ Let v_k be a vertex farthest from v_i such that $e_{\mu_i} = \delta_{\mu_i,\mu_k}$ and $e_{\gamma_i} = \delta_{\gamma_i,\gamma_k}$. Then by triangle inequality $e_{\mu_i} = \delta_{\mu_i,\mu_k} \leq \delta_{\mu_i,\mu_j} + \delta_{\mu_j,\mu_k}$ for any v_k of G. Therefore,

$e_{\mu_i} \leq \delta_{\mu_i,\mu_k} + \delta_{\mu_k,\mu_j}$ for any v_k of G.

That is, $e_{\mu_i} \leq \delta_{\mu_i,\mu_j} + e_{\mu_j}$. since $\delta_{\mu_k,\mu_j} \leq e_{\mu_j}$

Therefore, $e_{\mu_i} - e_{\mu_k} \leq \delta_{\mu_i,\mu_k}$.

Interchanging the roles of v_i and v_j, we get $e_{\mu_j} - e_{\mu_i} \leq \delta_{\mu_j,\mu_i}$, that is

$-\delta_{\mu_j,\mu_i} \leq e_{\mu_i} - e_{\mu_j}$.

Combining of the above result $-\delta_{\mu_j,\mu_i} \leq e_{\mu_i} - e_{\mu_j} \leq \delta_{\mu_j,\mu_i}$.

Similarly, $-\delta_{\gamma_j,\gamma_i} \leq e_{\gamma_i} - e_{\gamma_j} \leq \delta_{\gamma_j,\gamma_i}$. Hence, $|e_\mu(v_i) - e_\mu(v_j)| \leq \delta_{\mu_i,\mu_j}$ and

$|e_\gamma(v_i) - e_\gamma(v_j)| \leq \delta_{\gamma_i,\gamma_j}$.

Theorem 6.3.15. Let v_i and v_j be any two vertices in an IFT $G = (V, E)$. Then $\left|\delta_{\mu_i,\mu_k} - \delta_{\mu_k,\mu_j}\right| \leq \delta_{\mu_i,\mu_j}$, $\left|\delta_{\gamma_i,\gamma_k} - \delta_{\gamma_k,\mu_j}\right| \leq \delta_{\gamma_i,\gamma_j}$, for all v_k in V.

Proof:

Let v_i, v_j be any two vertices in V. Since $\delta(v_i, v_j)$ is a metric

$\delta_{\mu_i,\mu_j} \leq \delta_{\mu_i,\mu_k} + \delta_{\mu_k,\mu_j}$ and $\delta_{\gamma_i,\gamma_j} \leq \delta_{\gamma_i,\gamma_k} + \delta_{\gamma_k,\gamma_j}$

for all v_k in V. Also $\delta_{\mu_k,\mu_j} \leq \delta_{\mu_k,\mu_i} + \delta_{\mu_i,\mu_j}$ and $\delta_{\gamma_k,\gamma_j} \leq \delta_{\gamma_k,\gamma_i} + \delta_{\gamma_i,\gamma_j}$.

That is , $\delta_{\mu_i,\mu_j} \geq \delta_{\mu_k,\mu_i} - \delta_{\mu_k,\mu_j}$, $\delta_{\gamma_i,\gamma_j} \geq \delta_{\gamma_k,\gamma_i} - \delta_{\gamma_k,\gamma_j}$.

Combining the above results,

$\delta_{\mu_k,\mu_i} - \delta_{\mu_k,\mu_j} \leq \delta_{\mu_i,\mu_j} \leq \delta_{\mu_k,\mu_i} + \delta_{\mu_k,\mu_j}$ $\delta_{\gamma_k,\gamma_i} - \delta_{\gamma_k,\gamma_j} \leq \delta_{\gamma_i,\gamma_j} \leq \delta_{\gamma_k,\gamma_i} + \delta_{\gamma_k,\gamma_j}$.

That is, $-(\delta_{\mu_k,\mu_j} - \delta_{\mu_k,\mu_k}) \leq \delta_{\mu_i,\mu_j} \leq \delta_{\mu_k,\mu_i} + \delta_{\mu_k,\mu_j}$ and $-(\delta_{\gamma_k,\gamma_j} - \delta_{\gamma_k,\gamma_k}) \leq \delta_{\gamma_i,\gamma_j} \leq \delta_{\gamma_k,\gamma_i} + \delta_{\gamma_k,\gamma_j}$.

Therefore, δ_{μ_i,μ_j}, $\left|\delta_{\gamma_i,\gamma_k} - \delta_{\gamma_k,\mu_j}\right| \leq \delta_{\gamma_i,\gamma_j}$, for all v_k in V.

Theorem 6.3.16. Let $G = (V, E)$ be an IFT on $V \geq 3$. let $\in (H)$ be the maximum cardinality of end vertices in any spanning forest $H = (V', E')$ in G , then $d_c = o(G) - \in (H)$.

Proof:

Let H be spanning forest of G. Let $X = \{v_i \in V', v_i \text{is a end vertices of } H\}$. Clearly $V - X$ is a connected dominating set of G and the cardinality of $V - X$ is $o(G) - \in (H)$. Hence $d_c(G) \leq o(G) - \in (H)$.

Now, let S be a connected dominating set of G. Let H_S be any spanning forest of the induced subgraph $G[S]$. Since S is a connected dominating set G, for each v_i in $V - S$, there exists a vertex v_j in S such that $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$. Let H be the subgraph adding the vertices of $V - S$ and the edges $v_i v_j$ for each v_i in $V - S$. Clearly H is a spanning forest of G and $\in (H) \geq o(G) - d_c(G)$.

Hence $d_c(G) = o(G) - \in (H)$.

Theorem 6.3.17. Let $G = (V, E)$ be an IFT with $V \geq 3$. Suppose that $d(G - e) =$

$d(G)$, where e is an strong edge of G. Then for each strong edge $e = (v_i, v_j)$, there exists a dominating set D satisfying one of the following

(i) $v_i, v_j \in D$

(ii) $v_i, v_j \in V - D$

(iii) If $v_i \in D$ and $v_j \in V - D$, then there exists $v_k \in D - \{v_j\}$ such that $\mu_{jk} \geq \mu_{jk}^{\infty}$ and $\gamma_{jk} \geq \gamma_{jk}^{\infty}$.

Proof:

Suppose there is no dominating set D in G satisfying any of the statements (i),(ii), (iii). Then any dominating set D of G is not a dominating set of $G - e$. Further any dominating set of $G - e$ is a dominating set of G also. Hence, it follows that, $d(G - e) \neq d(G)$.

Theorem 6.3.18. In an IFT $G = (V, E)$ the cut vertices are dominating set of G.

Proof:

Let D be the dominating set of all dominating sets of G. Since every vertex in a spanning subgraph is either cut vertex or end vertex. Then $V - D$ is the set of all end vertices of G. Then, for each $v_i \in V - D$, there exists a strong neighbor $v_j \in D$. Hence each $v_i \in V - D$, is dominated by some $v_j \in D$. So D is a dominating set of G.

Theorem 6.3.19. If $G = (V, E)$ is an IFT, then G is not complete.

Proof:

Suppose G be a complete IFG. Let H be spanning subgraph of G. Then $\mu_{ij}^{\infty} = \mu_{ij}$ and $\gamma_{ij}^{\infty} = \gamma_{ij}$ for all v_i, v_j in V. Now G being a IFT, $\mu_{ij} < \mu_{ij}^{\infty}$ and $\gamma_{ij} < \gamma_{ij}^{\infty}$ for all v_i, v_j not in H, where$H$ be spanning subgraph of G. Thus $\mu_{ij}^{\infty} < \mu_{ij}$ and $\gamma_{ij}^{\infty} < \gamma_{ij}$, contradicting the definition of complete IFG.

6.4 Intuitionistic fuzzy tree center-based clustering algorithm

The objective of clustering is to classify the observations into groups such that the degree of association is high among the members of a group and is less among the members of other groups. Graph theoretical clustering is nothing but partitioning the graph based on qualitative aspects of the data. Most of the clustering methods group the data based on distance and similarity. Rosenfeld [86] introduced distance based clustering on fuzzy graphs. Xu and Wu [112] developed an intuitionistic fuzzy c-means algorithm to cluster IFSs, which is based on the well-known fuzzy c-means clustering method and the basic distance measures between IFSs such as the Hamming distance, normalized Hamming distance, Euclidean distance and normalized Euclidean distance. Zhang and Chen [114] defined the concept of intuitionistic fuzzy similarity matrix and presented a clustering method based on λ-cutting matrix. Zhong Wang et al. [116] presented a netting method to make cluster analysis of intuitionistic fuzzy sets. Zhao et al. [115] developed an intuitionistic fuzzy minimum spanning tree clustering algorithm to deal with intuitionistic fuzzy information.

In this section, a new clustering method namely, intuitionistic fuzzy tree center-based clustering method is proposed to classify the given crisp data set. The intuitionistic fuzzification of the data set is obtained by S-shaped intuitionistic fuzzification function. The classical similarity and distance measures are characterized by real numbers. The proposed distance measure is an intuitionistic fuzzy value. cluster center is not chosen randomly, but is obtained by using eccentricity concept in IFTs. The computation procedure of this method is comparatively easier. The proposed clustering algorithm is verified with a numerical data set.

Notations

Let $V = \{v_1, v_2, v_3 \dots v_n\}$ be the data set of n objects to be clustered . Let $A = \{A_1, A_2, A_3 \cdots, A_m\}$ is the set of m attributes for each object v_i. The data set is represented as a matrix $G = \left[v_i^p\right], i = 1, 2, 3 \dots, n, p = 1, 2, 3 \dots, m$. The columns (i) of the matrix G indicate the set of n objects and rows (p) represent as the number of numerical attributes of each data. The object v_i^p in the data matrix represents i^{th} object with p^{th} attribute.

The entries of the data matrix G are of the form $I_G = \left[\left\langle \mu_i^p, \gamma_i^p \right\rangle\right]_{n\times m}, i = 1, 2, 3, \dots, n$ $p = 1, 2, 3 \dots, m$ where $\left\langle \mu_i^p, \gamma_i^p \right\rangle$ represents the degree of membership and non-membership of i^{th} object with p^{th} attribute.

Definition 6.4.1. Let $D = (d_{ij})_{n\times n}$ be an intuitionistic fuzzy distance matrix, where $d_{ij} = \langle \mu_{ij}, \gamma_{ij} \rangle, i, j = 1, 2, 3, \dots n$. Then $(\lambda, \delta)_D = (\lambda, \delta)_{d_{ij}} = \left\langle \lambda_{\mu_{ij}}, \delta_{\gamma_{ij}} \right\rangle$ is called (λ, δ)-cutting matrix of D where (λ, δ) is the confidence level $0 \leq \lambda, \delta \leq$

$1, 0 \leq \lambda + \delta \leq 1$, and

$$(\lambda, \delta)_{d_{ij}} = \begin{cases} (1,0) & \text{if } \mu_{ij} \geq \lambda,\, \gamma_{ij} < \delta \\ (0,1) & \text{if } \mu_{ij} < \lambda \text{ and } \gamma_{ij} \geq \delta \end{cases} \tag{6.1}$$

Definition 6.4.2. [114] Let $\alpha, \beta \in X_{1\times n}$, where $X_{1\times n}$ denotes the set of intuitionistic fuzzy vectors. Then $(\alpha, \beta) = (max\left\{min\left\{\mu_{\alpha i}, \mu_{\beta i}\right\}\right\}, min\left\{max\left\{\gamma_{\alpha i}, \gamma_{\beta i}\right\}\right\})$ is called the inner product of α and β.

The step by step procedure of proposed intuitionistic fuzzy tree center based algorithm is described here.

Algorithm

Step 1: Consider, $V = (v_1, v_2, v_3 \cdots, v_n)$, the set of n objects and $A = \{A_1, A_2, A_3 \cdots, A_m\}$ is a set of m attributes in a data set. Form the data matrix G.

Step 2: The intuitioistic fuzzification for the data set of n objects is done as follows:

The degree of membership μ_i^p, is calculated using

$$\mu_i^p = \begin{cases} 0 & \text{if} \quad v_i^p \leq a \\ 2\left(\frac{v_i^p - a}{b-a}\right)^2 - \epsilon & \text{if} \quad a < v_i^p \leq \frac{a+b}{2} \\ 1 - 2\left(\frac{v_i^p - b}{b-a}\right)^2 - \epsilon & \text{if} \quad \frac{a+b}{2} < v_i^p < b \\ 1 - \epsilon & \text{if} \quad v_i^p \geq b \end{cases} \tag{6.2}$$

The degree of non-membership γ_i^p is calculated by

$$\gamma_i^p = \begin{cases} 1-\epsilon & \text{if} \quad v_i^p \leq a \\ 1-2\left(\frac{v_i^p-a}{b-a}\right)^2 & \text{if} \quad a < v_i^p \leq \frac{a+b}{2} \\ 2\left(\frac{v_i^p-b}{b-a}\right)^2 & \text{if} \quad \frac{a+b}{2} < v_i^p < b \\ 0 & \text{if} \quad v_i^p \geq b \end{cases} \tag{6.3}$$

where a, b , c are arbitrary constants.

Step 3: Calculate the distance between two objects using the formula

$$d(v_i, v_j) = \begin{cases} \langle 0, 1 \rangle, \quad i = j \\ \left\langle 1 - \frac{1}{m} \sum_{p=1}^{m} \left|\mu_i^p - \mu_j^p\right|, \frac{1}{m} \sum_{p=1}^{m} \left|\gamma_i^p - \gamma_j^p\right| \right\rangle, \\ i \neq j, i, j = 1, 2, \cdots, n. \end{cases} \tag{6.4}$$

Form the IF distance matrix $D = d(v_i, v_j)_{n \times n}$

Step 4: Draw the IFT $G = (V, E)$ with n vertices associated with the objects v_i in the data set V to be clustered. The distance $d(v_i, v_j)$ is treated as the membership and non-membership values of the edges.

Step 5: Compute the eccentricity of each data object $v_i \in V$ by using the formula $e(v_i) = \langle e_{\mu_i}, e_{\gamma_i} \rangle = \langle max(d_\mu(v_i, v_j), max(d_\gamma(v_i, v_j)) \rangle$.

Step 6: Calculate the radius as $r(G) = \langle r_\mu(G), r_\gamma(G) \rangle$, where, $r_\mu(G) =$ min $\langle e_{\mu_i} : v_i \in V \rangle$ and $r_\gamma(G) =$ min $\{e_{\gamma_i} : v_i \in V\}$.

Step 7: Compute center of G

[i] $e(v_i)$, if $r_\mu(G) = e_{\mu_i}$ and $r_\gamma(G) = e_{\gamma_i}$ or the corresponding e_{μ_j} of $e_{\gamma_j} = r_\gamma(G)$ is less than e_{μ_i}.

[ii] $e(v_j)$, if $r_\mu(G) = e_{\mu_i}$ and $r_\gamma(G) = e_{\gamma_j}$, the corresponding e_{μ_j} of $e_{\gamma_j} = r_\gamma(G)$ is greater than or equal to e_{μ_i}.

Step 8: Treat the center $e(v_i)$ obtained in Step 7 as $(\lambda, \delta)-$cut.

Step 9: Calculate the (λ, δ)-cutting matrix on $(\lambda, \delta)_{d(v_i,v_j)}$ using Definition 6.4.1.

Step 10: Calculate the inner products of the column vectors of the $(\lambda, \delta)_{d(v_i,v_j)}$-cutting matrix. Then the objects are clustered based on the inner product values $(1, 1)$ or $(1, 0)$ using Definition 6.4.2.

Step 11: Go to Step 6, repeat the process to get clusters.

6.5 Experimental analysis

The algorithm has been implemented and tested with datasets available in the University of Cologne, Germany [119]. The data sets contain the nutrients in 27 different kinds of meat, fish or fowl with five attributes as food energy, protein, fat, calcium and iron. The data set is divided into five disjoint subsets. The information about the data set with 5 attributes is shown in Table 1. The intuitionistic fuzzification of the data set is presented in Table 2.

Step wise algorithm

The steps involved to cluster the numerical data set with 27 nutrients and 5 attributes.

Step 1. Consider the data set is given in Table 1 to produce cluster.

Step 2. Compute the degree of membership and non-membership values for the given data set using S-shape intuitionistic fuzzification function using equation 2, 3 are given in Table 2.

Step 3. Obtain IFT, by treating 27 nutrients as vertices $v_i, \cdots, v_{27}$. The distance between v_i and v_j is treated as the weight of e_{ij}. Calculate distance between the objects v_i, v_j

Step 4. The eccentricities are given by

$e(v_1) = \langle 0.9960, 0.6559\rangle$, $e(v_2) = \langle 0.9358, 0.4994\rangle$,

$e(v_3) = \langle 0.8847, 0.6653\rangle$, $e(v_4) = \langle 0.9771, 0.6739\rangle$,

$e(v_5) = \langle 0.9614, 0.4516\rangle$, $e(v_6) = \langle 0.8838, 0.5522\rangle$,

$e(v_7) = \langle 0.9949, 0.4859\rangle$, $e(v_8) = \langle 0.8329, 0.6653\rangle$,

$e(v_9) = \langle 0.9358, 0.5393\rangle$, $e(v_{10}) = \langle 0.9266, 0.5928\rangle$,

$e(v_{11}) = \langle 0.9360, 0.6599\rangle$, $e(v_{12}) = \langle 0.9847, 0.6513\rangle$,

$e(v_{13}) = \langle 0.9847, 0.6666\rangle$, $e(v_{14}) = \langle 0.9353, 0.4003\rangle$,

$e(v_{15}) = \langle 0.9439, 0.4710\rangle$, $e(v_{16}) = \langle 0.9486, 0.5089\rangle$,

$e(v_{17}) = \langle 0.9669, 0.6739\rangle$, $e(v_{18}) = \langle 0.9669, 0.6717\rangle$,

$e(v_{19}) = \langle 0.9393, 0.5115\rangle$, $e(v_{20}) = \langle 0.9393, 0.4524\rangle$,

$e(v_{21}) = \langle 0.9353, 0.4421\rangle$, $e(v_{22}) = \langle 0.9298, 0.4815\rangle$,

$e(v_{23}) = \langle 0.9295, 0.3819\rangle$, $e(v_{24}) = \langle 0.9298, 0.5129\rangle$,

$e(v_{25}) = \langle 0.7802, 0.6589\rangle$, $e(v_{26}) = \langle 0.9949, 0.49907\rangle$,

$e(v_{27}) = \langle 0.9106, 0.5545\rangle$.

Step 5. The radius is calculated as $\langle 0.7802, 0.3819\rangle$

Step 6. The center is e_{23}. Treat e_{23} as (λ, δ)-cut in the distance matrix $d(v_i, v_j)$, the clusters are obtained.

If $(\lambda, \delta) = \langle 0.9295, 0.3819 \rangle$, then the objects $v_i, 1 = 1, 2 \ldots, 27$ are fall into the following eighteen categories:

$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_3\}$,$\{v_5, v_7, v_{26}\}$,$\{v_{17}, v_{18}\}$,$\{v_{22}, v_{24}\}$,$\{v_2\}$, $\{v_3\}$, $\{v_6\}$, $\{v_8\}$, $\{v_9\}$, $\{v_{14}\}$, $\{v_{15}\}$, $\{v_{16}\}$, $\{v_{19}\}$,$\{v_{20}\}$, $\{v_{21}\}$, $\{v_{23}\}$, $\{v_{25}\}$, $\{v_{27}\}$

If $(\lambda, \delta) = \langle 0.9266, 0.5928 \rangle$, then the objects $v_i, 1 = 1, 2 \ldots, 27$ are fall into the following seventeen categories:

$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_3\}$,$\{v_5, v_7, v_{16}, v_{26}\}$,$\{v_{17}, v_{18}\}$, $\{v_{22}, v_{24}\}$, $\{v_2\}$, $\{v_3\}$, $\{v_6\}$, $\{v_8\}$, $\{v_9\}$, $\{v_{14}\}$, $\{v_{15}\}$, $\{v_{19}\}$, $\{v_{20}\}$, $\{v_{21}\}$, $\{v_{23}\}$, $\{v_{25}\}$, $\{v_{27}\}$

If $(\lambda, \delta) = \langle 0.9106, 0.5545 \rangle$, then the objects $v_i, 1 = 1, 2 \ldots, 27$ are fall into the following sixteen categories:

$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_3\}$,$\{v_5, v_7, v_{16}, v_{26}\}$,$\{v_{17}, v_{18}\}$, $\{v_{22}, v_{24}\}$, $\{v_2, v_9\}$, $\{v_3\}$, $\{v_6\}$, $\{v_8\}$, $\{v_{14}\}$, $\{v_{15}\}$, $\{v_{19}\}$, $\{v_{20}\}$, $\{v_{21}\}$, $\{v_{23}\}$, $\{v_{25}\}$, $\{v_{27}\}$.

Results and Discussion

There are many clustering algorithm existing in the literature. The results of the proposed algorithm is compared with only two algorithm namely Zhang et al. [114] and netting method by Z.Wang et al.[116]. The derived results are compared and presented in Table 3. Though, all the three methods produce same clusters, the presented IFT center-based algorithm reduces the complexity of calculations in forming equivalence matrix, which ultimately reduces the running time of the algorithm.

Table 1: Data set with 5 attributes and 27 objects

Objects	**Name of the item**	**Food Energy (Calories)** A_1	**Protein (Grams)** A_2	**Fat (Grams)** A_3	**Calcium (Mgs)** A_4	**Iron (Mgs)** A_5
v_1	Beef braised	340	20	28	9	2.6
v_2	Hamburger	245	21	17	9	2.7
v_3	Beef roast	420	15	39	7	2.0
v_4	Beef steak	375	19	32	9	2.6
v_5	Beef canned	180	22	10	17	1.4
v_6	Chicken broiled	115	20	3	8	3.7
v_7	Chicken canned	170	25	7	12	1.5
v_8	Beef Heart	160	26	5	14	6.9
v_9	Lamp leg roast	265	20	20	9	2.6
v_{10}	Lamb shoulder roast	300	18	25	9	2.3
v_{11}	Smoked ham	340	20	28	9	2.5
v_{12}	Pork roast	340	19	29	9	2.5
v_{13}	Pork simmered	355	19	30	9	2.4
v_{14}	Beef tongue	205	18	14	7	2.5
v_{15}	Veal cutlet	185	23	9	9	2.7
v_{16}	Bluefish baked	135	22	4	25	0.6
v_{17}	Clams raw	70	11	1	82	6.0
v_{18}	Clams canned	45	7	1	74	5.4
v_{19}	Crab meat canned	90	14	2	38	0.8
v_{20}	Haddock fried	135	16	5	15	0.5
v_{21}	Mackerel broiled	200	19	13	5	1.0
v_{22}	Mackerel canned	155	16	9	157	1.8
v_{23}	Perch fried	195	16	11	14	1.3
v_{24}	Salmon canned	120	17	5	159	0.7
v_{25}	Sardines canned	180	22	9	367	2.5
v_{26}	Tuna canned	170	25	7	7	1.2
v_{27}	Shrimp canned	110	23	1	98	2.6

Table 2: Intuitionistic fuzzification of Data set

Objects	**Name the item**	**Food Energy (Calories)** A_1	**Protein (Grams)** A_2	**Fat (Grams)** A_3	**Calcium (Milli Grams)** A_4	**Iron (Milli Grams)** A_5
v_1	Beef braised	$\langle 0.9089, 0.0910\rangle$	$\langle 0.7806, 0.1994\rangle$	$\langle 0.8321, 0.1676\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.2150, 0.7847\rangle$
v_2	Hamburger	$\langle 0.5643, 0.4356\rangle$	$\langle 0.8415, 0.1386\rangle$	$\langle 0.3542, 0.6454\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.2360, 0.7637\rangle$
v_3	Beef roast	$\langle 0.999, 0\rangle$	$\langle 0.3346, 0.6454\rangle$	$\langle 0.9997, 0\rangle$	$\langle 0.0005, 0.9999\rangle$	$\langle 0.1096, 0.8901\rangle$
v_4	Beef steak	$\langle 0.9711, 0.0288\rangle$	$\langle 0.7085, 0.2715\rangle$	$\langle 0.9318, 0.0679\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.2150, 0.7847\rangle$
v_5	Beef canned	$\langle 0.2591, 0.7408\rangle$	$\langle 0.8914, 0.0886\rangle$	$\langle 0.1119, 0.8878\rangle$	$\langle 0.0021, 0.9978\rangle$	$\langle 0.0393, 0.9604\rangle$
v_6	Chicken broiled	$\langle 0.0696, 0.9303\rangle$	$\langle 0.7806, 0.1994\rangle$	$\langle 0.0052, 0.9944\rangle$	$\langle 0.0001, 0.9999\rangle$	$\langle 0.4997, 0.500\rangle$
v_7	Chicken canned	$\langle 0.2221, 0.7777\rangle$	$\langle 0.9744, 0.005\rangle$	$\langle 0.0496, 0.9501\rangle$	$\langle 0.0007, 0.9993\rangle$	$\langle 0.0485, 0.9512\rangle$
v_8	Beef Heart	$\langle 0.1880, 0.8119\rangle$	$\langle 0.9800, 0\rangle$	$\langle 0.0219, 0.9778\rangle$	$\langle 0.0012, 0.9988\rangle$	$\langle 0.9997, 0\rangle$
v_9	Lamp leg roast	$\langle 0.6582, 0.3417\rangle$	$\langle 0.7806, 0.1994\rangle$	$\langle 0.4997, 0.500\rangle$	$\langle 0.0002, 0.9997\rangle$	$\langle 0.2150, 0.7847\rangle$
v_{10}	Lamb shoulder roast	$\langle 0.7951, 0.2048\rangle$	$\langle 0.6254, 0.3546\rangle$	$\langle 0.7282, 0.2715\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.1579, 0.8418\rangle$
v_{11}	Smoked ham	$\langle 0.9089, 0.0910\rangle$	$\langle 0.7806, 0.1994\rangle$	$\langle 0.8321, 0.1676\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.1950, 0.8047\rangle$
v_{12}	Pork roast	$\langle 0.9089, 0.0910\rangle$	$\langle 0.7085, 0.2714\rangle$	$\langle 0.8612, 0.1385\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.1950, 0.8047\rangle$
v_{13}	Pork simmered	$\langle 0.9398, 0.0601\rangle$	$\langle 0.7085, 0.2715\rangle$	$\langle 0.8875, 0.1122\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.1760, 0.8237\rangle$
v_{14}	Beef tongue	$\langle 0.3640, 0.359\rangle$	$\langle 0.6254, 0.3546\rangle$	$\langle 0.2338, 0.7659\rangle$	$\langle 0, 0.9999\rangle$	$\langle 0.1950, 0.8047\rangle$
v_{15}	Veal cutlet	$\langle 0.2787, 0.7212\rangle$	$\langle 0.9301, 0.0499\rangle$	$\langle 0.0883, 0.9114\rangle$	$\langle 0.0002, 0.9998\rangle$	$\langle 0.2360, 0.7637\rangle$
v_{16}	Bluefish baked	$\langle 0.1151, 0.8848\rangle$	$\langle 0.8914, 0.0886\rangle$	$\langle 0.0122, 0.9875\rangle$	$\langle 0.0060, 0.9939\rangle$	$\langle 0.0001, 0.9995\rangle$
v_{17}	Clams raw	$\langle 0.0088, 0.9911\rangle$	$\langle 0.0686, 0.9114\rangle$	$\langle 0, 0.9997\rangle$	$\langle 0.0945, 0.9095\rangle$	$\langle 0.9601, 0.0395\rangle$
v_{18}	Clams canned	$\langle 0, 0.9999\rangle$	$\langle 0, 0.9800\rangle$	$\langle 0, 0.9997\rangle$	$\langle 0.0727, 0.9273\rangle$	$\langle 0.8898, 0.1099\rangle$
v_{19}	Crab meat canned	$\langle 0.0287, 0.9712\rangle$	$\langle 0.2515, 0.7285\rangle$	$\langle 0.0010, 0.9986\rangle$	$\langle 0.0166, 0.9833\rangle$	$\langle 0.0041, 0.0056\rangle$
v_{20}	Haddock fried	$\langle 0.1151, 0.8848\rangle$	$\langle 0.4287, 0.5512\rangle$	$\langle 0.0218, 0.9778\rangle$	$\langle 0.0015, 0.9984\rangle$	$\langle 0, 0.9997\rangle$
v_{21}	Mackerel broiled	$\langle 0.3415, 0.6583\rangle$	$\langle 0.7085, 0.2714\rangle$	$\langle 0.1991, 0.8005\rangle$	$\langle 0, 0.9999\rangle$	$\langle 0.0119, 0.9877\rangle$
v_{22}	Mackerel canned	$\langle 0.1720, 0.8279\rangle$	$\langle 0.4287, 0.5512\rangle$	$\langle 0.0883, 0.9113\rangle$	$\langle 0.3526, 0.6473\rangle$	$\langle 0.0822, 0.9174\rangle$
v_{23}	Perch fried	$\langle 0.3199, 0.6800\rangle$	$\langle 0.4287, 0.5512\rangle$	$\langle 0.1382, 0.8614\rangle$	$\langle 0.0012, 0.9987\rangle$	$\langle 0.0310, 0.9688\rangle$
v_{24}	Salmon canned	$\langle 0.0799, 0.92\rangle$	$\langle 0.5312, 0.4487\rangle$	$\langle 0.0218, 0.9778\rangle$	$\langle 0.3619, 0.6380\rangle$	$\langle 0.0016, 0.9980\rangle$
v_{25}	Sardines canned	$\langle 0.2591, 0.7408\rangle$	$\langle 0.8913, 0.0886\rangle$	$\langle 0.0883, 0.9113\rangle$	$\langle 0.9999, 0\rangle$	$\langle 0.1950, 0.8046\rangle$
v_{26}	Tuna canned	$\langle 0.2221, 0.7777\rangle$	$\langle 0.9744, 0.0055\rangle$	$\langle 0.0495, 0.9501\rangle$	$\langle 0, 0.9999\rangle$	$\langle 0.0236, 0.9760\rangle$
v_{27}	Shrimp canned	$\langle 0.0599, 0.9399\rangle$	$\langle 0.9301, 0.0498\rangle$	$\langle 0, 0.9997\rangle$	$\langle 0.1319, 0.8679\rangle$	$\langle 0.2150, 0.7846\rangle$

Table 3: Comparisons of the derived results

Cluster	The results derived by center of IFT method	Zhang et.al method [[114]]	Zhong Wang et.al method[[116]]
1	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23} v_{24}, v_{25}, v_{26}, v_{27}\}$,	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23} v_{24}, v_{25}, v_{26}, v_{27}\}$,	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23} v_{24}, v_{25}, v_{26}, v_{27}\}$,
2	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$
3	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$	-	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$
4	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$
5	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}\{v_{22}, v_{24}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}\{v_{22}, v_{24}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}\{v_{22}, v_{24}\}$
6	$\{v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$	$\{v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$, $\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$	$\{v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$
7	$\{v_1, v_2, v_4, v_5, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$, $\{v_6\}$	$\{v_1, v_2, v_4, v_5, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$, $\{v_6\}$	$\{v_1, v_2, v_4, v_5, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$, $\{v_6\}$
8	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$,$\{v_2, v_5, v_7, v_9, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$, $\{v_6\}$	-	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$,$\{v_2, v_5, v_7, v_9, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$, $\{v_{17}, v_{18}\}$,$\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$, $\{v_6\}$
9	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{19}, v_{20}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$
10	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{21}, v_{23}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$
11	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$, $\{v_{25}\}$, $\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$,$\{v_{23}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$, $\{v_{25}\}$, $\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$,$\{v_{23}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_5, v_7, v_{14}, v_{15}, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$,$\{v_8\}$, $\{v_{25}\}$, $\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$,$\{v_{23}\}$
12	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}\{v_5, v_7, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$,$\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}, v_{15}\}$	-	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$,$\{v_5, v_7, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$,$\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}, v_{15}\}$
13	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$,$\{v_5, v_7, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$,$\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}\}$, $\{v_{15}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$,$\{v_5, v_7, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$,$\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}\}$, $\{v_{15}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}$,$\{v_5, v_7, v_{16}, v_{21}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$, $\{v_2, v_9\}$,$\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}\}$, $\{v_{15}\}$
14	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}\{v_5, v_7, v_{16}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}\}$, $\{v_{15}\}$,$\{v_{21}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}\{v_5, v_7, v_{16}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}\}$, $\{v_{15}\}$,$\{v_{21}\}$	$\{v_1, v_4, v_{10}, v_{11}, v_{12}, v_{13}\}\{v_5, v_7, v_{16}, v_{26}, v_{27}\}$,$\{v_{17}, v_{18}\}$, $\{v_8\}$,$\{v_{25}\}$,$\{v_{22}, v_{24}\}$,$\{v_3\}$,$\{v_6\}$,$\{v_2, v_9\}$, $\{v_{19}, v_{20}\}$,$\{v_{23}\}$, $\{v_{14}\}$, $\{v_{15}\}$,$\{v_{21}\}$

Table 3: Comparisons of the derived results

Cluster	The results derived by center of IFT method	Zhang et.al method [[114]]	Zhong Wang et.al method[[116]]
15	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2,v_9\},\{v_{19},v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$	-	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2,v_9\},\{v_{19},v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$
16	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2,v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2,v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2,v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$
17	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{16},v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\}$
18	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$
19	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{10},v_{11},v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$
20	$\{v_1,v_4,v_{10},v_{11}\},\{v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	-	$\{v_1,v_4,v_{10},v_{11}\},\{v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$
21	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5,v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$
22	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17},v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$
23	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$	-	$\{v_1,v_4,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\}$
24	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12},v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$
25	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12}\},\{v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$	-	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12}\},\{v_{13}\},\{v_5\},\{v_7,v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$
26	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12}\},\{v_{13}\},\{v_5\},\{v_7\},\{v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12}\},\{v_{13}\},\{v_5\},\{v_7\},\{v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$	$\{v_1,v_{11}\},\{v_{10}\},\{v_{12}\},\{v_{13}\},\{v_5\},\{v_7\},\{v_{26}\},\{v_{17}\},\{v_{18}\},\{v_8\},\{v_{25}\},\{v_{22}\},\{v_{24}\},\{v_3\},\{v_6\},\{v_2\},\{v_9\},\{v_{19}\},\{v_{20}\},\{v_{23}\},\{v_{14}\},\{v_{15}\},\{v_{21}\},\{v_{27}\},\{v_{16}\},\{v_4\}$

Chapter 7

Chromatic values of intuitionistic fuzzy directed hypergraph colorings

7.1 Introduction

Fuzzy sets (FSs) introduced by L.A.Zadeh in 1965 [110] are generalization of crisp sets. K.T.Atanassov introduced the concept of intuitionistic fuzzy sets (IFSs) in 1999 [9] as an extension of FSs. These sets include not only the membership of the set but also the non-membership of the set along with degree of uncertainity. In order to expand the application base, the notion of graph was generalized to that of a hypergraph. In 1976, Berge [28] introduced the concepts of graph and hypergraph. This paper contains a few extensions of concepts in fuzzy hypergraphs by John N. Mordeson and Premchand S. Nair [65].

The chapter has been organised as follows: Section 2 deals with the definitions of fuzzy hypergraph, intuitionistic fuzzy hypergraph, IFDHG and the notations used in this paper. In section 3 and 4, a study is made on core aggregate of IFDHG, conservative $\mathcal{K}-$ coloring of intuitionistic fuzzy directed hypergraph, chromatic values of intuitionistic fuzzy colorings, elementary center of intuitionistic fuzzy coloring, f-chromatic value of intuitionistic fuzzy coloring, intersecting IFDHG, $\mathcal{K}-$ intersecting IFDHG, strongly intersecting IFDHG. Some properties of the newly proposed hypergraph concepts are also discussed. Section 5 concludes the chapter.

7.2 Notations and Preliminaries

The notations used in this work are listed below:

$H = (V, E)$	- IFDHG with vertex set V and edge set E
$\mu(v_i), \nu(v_i)$	- degrees of membership and non-membership of the vertex
μ_{ij}, ν_{ij}	- degrees of membership and non-membership of the edges
$\mu_{ij}(v_i), \nu_{ij}(v_i)$	- degrees of membership and non-membership of the edges containing v_i
$h(H)$	- height of a hypergraph H
$F(H)$	- Fundamental sequence of H
$C(H)$	- Core set of H
$H^{(r_i,s_i)}$	- (r_i, s_i) - level intuitionistic fuzzy hypergraph
$\mathcal{IF}_p(v)$	- intuitionistic fuzzy power set of V.

In this section, definitions of intuitionistic fuzzy hypergraph, IFDHG are dealt with.

Definition 7.2.1. [9] The five Cartesian products of two IFSs V_1, V_2 of V over E is defined as

$$V_1 \times_1 V_2 = \{\langle (v_1, v_2), \mu_1.\mu_2, \nu_1.\nu_1\rangle | v_1 \in V_1, v_2 \in V_2\},$$
$$V_1 \times_2 V_2 = \{\langle (v_1, v_2), \mu_1 + \mu_2 - \mu_1\mu_2, \nu_1.\nu_2\rangle | v_1 \in V_1, v_2 \in V_2\},$$
$$V_1 \times_3 V_2 = \{\langle (v_1, v_2), \mu_1.\mu_2, \nu_1 + \nu_2 - \nu_1.\nu_2\rangle | v_1 \in V_1, v_2 \in V_2\},$$
$$V_1 \times_4 V_2 = \{\langle (v_1, v_2), \min(\mu_1, \mu_2), \max(\nu_1, \nu_2)\rangle | v_1 \in V_1, v_2 \in V_2\},$$
$$V_1 \times_5 V_2 = \{\langle (v_1, v_2), \max(\mu_1, \mu_2), \min(\nu_1, \nu_2)\rangle | v_1 \in V_1, v_2 \in V_2\},$$
$$V_1 \times_6 V_2 = \{\langle (v_1, v_2), \tfrac{\mu_1+\mu_2}{2}, \tfrac{\nu_1+\nu_2}{2}\rangle | v_1 \in V_1, v_2 \in V_2\}.$$

It must be noted that $v_i \times_s v_j$ is an IFS, where $s = 1, 2, 3, 4, 5, 6$.

Definition 7.2.2. [9] Let E be the fixed set and $V = \{\langle v_i, \mu_i(v_i), \nu_i(v_i)\rangle | v_i \in V\}$ be an IFS. Six types of cartesian products of n subsets[1] $V_1, V_2, \cdots, V_n$ of V over E are defined as

$$\begin{aligned} V_1 \times_1 V_2 \times_1 V_3 ... \times_1 V_n = \{\langle (v_1, v_2, \cdots, v_n), \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \nu_i\rangle \\ | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\}, \\ V_{i_1} \times_2 V_{i_2} \times_2 V_{i_3} ... \times_2 V_{i_n} = \{\langle (v_1, v_2, \cdots, v_n), \sum_{i=1}^{n} \mu_i - \sum_{i \neq j} \mu_i\mu_j + \\ \sum_{i \neq j \neq k} \mu_i\mu_j\mu_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \cdots \neq n} \mu_i\mu_j\mu_k \cdots \mu_n + \\ (-1)^{n-1} \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \nu_i\rangle | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\} \end{aligned}$$

[1] subsets - crisp sense

$$V_{i_1} \times_3 V_{i_2} \times_3 V_{i_3} ... \times_3 V_{i_n} = \{\langle (v_1, v_2, \cdots, v_n), \prod_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \nu_i - \sum_{i \neq j} \nu_i \nu_j + \sum_{i \neq j \neq k} \nu_i \nu_j \nu_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \cdots \neq n} \nu_i \nu_j \nu_k \cdots \nu_n + (-1)^{n-1} \prod_{i=1}^{n} \nu_i \rangle | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\}$$

$$V_1 \times_4 V_2 \times_4 V_3 ... \times_4 V_n = \{\langle (v_1, v_2, \cdots, v_n), \min(\mu_1, \mu_2, \cdots, \mu_n), \max(\nu_1, \nu_2, \cdots, \nu_n) \rangle | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\}$$

$$V_1 \times_5 V_2 \times_5 V_3 ... \times_5 V_n = \{\langle (v_1, v_2, \cdots, v_n), \max(\mu_1, \mu_2, \cdots, \mu_n), \min(\nu_1, \nu_2, \cdots, \nu_n) \rangle | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\}.$$

$$V_1 \times_6 V_2 \times_6 V_3 ... \times_6 V_n = \{\langle (v_1, v_2, \cdots, v_n), \frac{\sum_{i=1}^{n} \mu_i}{n}, \frac{\sum_{i=1}^{n} \nu_i}{n} \rangle | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\}.$$

It must be noted that $v_i \times_s v_j$ is an IFS, where $s = 1, 2, 3, 4, 5, 6$.

Definition 7.2.3. [53] An *intuitionistic fuzzy graph (IFG)* is of the form $G = \langle V, E \rangle$ where (i)$V = \{v_1, v_2, ...v_n\}$ such that $\mu_i : V \to [0, 1]$ and $\nu_i : V \to [0, 1]$ denote the degrees of membership and non-membership of the vertex $v_i \in V$ respectively and

$$0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1 \tag{7.1}$$

for every $v_i \in V, i = 1, 2, ..., n$(ii) $E \subseteq V \times V$ where $\mu_{ij} : V \times V \to [0, 1]$ and $\nu_{ij} : V \times V \to [0, 1]$ are such that

$$\mu_{ij} \leq \mu_i \oslash \mu_j \tag{7.2}$$

$$\nu_{ij} \leq \nu_i \oslash \nu_j \tag{7.3}$$

and

$$0 \leq \mu_{ij} + \nu_{ij} \leq 1 \tag{7.4}$$

where μ_{ij} and ν_{ij} are the degrees of membership and non-membership of the edge (v_i, v_j); the values of $\mu_i \oslash \mu_j$ and $\nu_i \oslash \nu_j$ can be determined by one of the cartesian products $\times_s$, $s = 1, 2, ..., 6$ for all i and j given in Definition 7.2.2.

Note:

Throughout this paper, it is assumed that the fifth Cartesian product in Definition 7.2.2

$$\begin{aligned} V_1 \times_5 V_2 \times_5 V_3 ... \times_5 V_n = \{\langle (v_1, v_2, \cdots, v_n), \max(\mu_1, \mu_2, \cdots, \mu_n), \\ \min(\nu_1, \nu_2, \cdots, \nu_n) \rangle | v_1 \in V_1, v_2 \in V_2, \cdots, v_n \in V_n\}. \end{aligned}$$

is used to determine the degrees of membership μ_{ij} and non-membership ν_{ij} of the edge e_{ij}.

Definition 7.2.4. [79] An *intuitionistic fuzzy hypergraph* (IFHG) is an ordered pair $H = (V, E)$ where

(i) $V = \{v_1, v_2, ..., v_n\}$, is a finite set of intuitionistic fuzzy vertices,

(ii) $E = \{E_1, E_2, ..., E_m\}$ is a family of crisp subsets of V,

(iii) $E_j = \{(v_i, \mu_j(v_i), \nu_j(v_j)) : \mu_j(v_i), \nu_j(v_i) \geq 0$ and $\mu_j(x_i) + \nu_j(x_i) \leq 1\}, j = 1, 2, ..., m$,

(iv) $E_j \neq \phi, j = 1, 2, ..., m$,

(v) $\bigcup_j supp(E_j) = V, j = 1, 2, ..., m$.

Here, the hyperedges E_j are crisp sets of intuitionistic fuzzy vertices, $\mu_j(v_i)$ and

$\nu_j(v_i)$ denote the degrees of membership and non-membership of vertex v_i to edge E_j. Thus, the elements of the incidence matrix of IFHG are of the form $(v_{ij}, \mu_j(v_i), \nu_j(v_j))$. The sets (V, E) are crisp sets.

Notations:

1. Hereafter,$\langle \mu(v_i), \nu(v_i) \rangle$ or simply $\langle \mu_i, \nu_i \rangle$ denote the degrees of membership and non-membership of the vertex $v_i \in V$, such that $0 \leq \mu_i + \nu_i \leq 1$.
2. $\langle \mu(v_{ij}), \nu(v_{ij}) \rangle$ or simply $\langle \mu_{ij}, \nu_{ij} \rangle$ denote the degrees of membership and non-membership of the edge $(v_i, v_j) \in V \times V$, such that $0 \leq \mu_{ij} + \nu_{ij} \leq 1$. Also μ_{ij} is the degrees of membership of i^{th} vertex in j^{th} edge and ν_{ij} is the degrees of non-membership of i^{th} vertex in j^{th} edge.

Note:

The support of an IFS V in E is denoted by *supp*$(E_j) = \{v_i / \mu_{ij}(v_i) > 0$ and $\nu_{ij}(v_i) > 0\}$.

Definition 7.2.5. [78] An *intuitionistic fuzzy directed hypergraph* (IFDHG) H is a pair (V, E), where V is a non - empty set of vertices and E is a set of intuitionistic fuzzy hyperarcs; an intuitionistic fuzzy hyperarc $E_i \in E$ is defined as a pair $(t(E_i), h(E_i))$, where $t(E_i) \subset V$, with $t(E_i) \neq \phi$, is its tail, and $h(E_i) \in V - t(E_i)$ is its head. A vertex s is said to be a *source vertex* in H if $h(E_i) \neq s$, for every $E_i \in E$. A vertex d is said to be a *destination vertex* in H if $d \neq t(E_i)$, for every $E_i \in E$.

Definition 7.2.6. [67] Let H be an IFDHG, for $0 < (r_i, s_i) \leq h(H)$, let $H^{r_i,s_i} = (V^{r_i,s_i}, E^{r_i,s_i})$ be the (r_i, s_i) - level intuitionistic fuzzy directed hypergraph of H. The sequence of real numbers $\{r_1, r_2, ..., r_n; s_1, s_2, ..., s_n\}$, such that $0 \leq r_i \leq$

$h_\mu(H)$ and $0 \leq s_i \leq h_\nu(H)$, satisfying the properties:

(i) If $r_1 < \alpha \leq 1$ and $0 \leq \beta < s_1$ then $E^{\alpha,\beta} = \phi$,

(ii) If $r_{i+1} \leq \alpha \leq r_i$; $s_i \leq \beta \leq s_{i+1}$ then $E^{\alpha,\beta} = E^{r_i,s_i}$,

(iii) $E^{r_i,s_i} \sqsubset E^{r_{i+1},s_{i+1}}$

is called the *fundamental sequence of* H, and is denoted by $F(H)$.

The core set of H is denoted by $C(H)$ and is defined by $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$.

The corresponding set of (r_i, s_i) - level hypergraphs $H^{r_1,s_1} \subset H^{r_2,s_2} \subset .... \subset H^{r_n,s_n}$ is called the H *induced fundamental sequence* and is denoted by *I(H).* The (r_n, s_n) level is called the *support level* of H and the H^{r_n,s_n} is called the *support* of H.

Definition 7.2.7. [67] Let H be an IFDHG and $C(H) = \{H^{r_1 s_1}, H^{r_2,s_2}, ...H^{r_n,s_n}\}$. H is said to be *ordered* if $C(H)$ is ordered. That is $H^{r_1,s_1} \subset H^{r_2,s_2} \subset ... \subset H^{r_n,s_n}$. The intuitionistic fuzzy directed hypergraph is said to be *simply ordered* if the sequence $\{H^{r_i,s_i}/i = 1, 2, 3..., n\}$ is simply ordered, that is if it is ordered and if whenever $E \in H^{r_{i+1},s_{i+1}} - H^{r_i,s_i}$ then $E \not\subseteq H^{r_i,s_i}$.

Definition 7.2.8. A *minimal intuitionistic fuzzy transversal* T for H is a transversal of H with the property that if $T_1 \subset T$, then T_1 is not an intuitionistic fuzzy transversal of H.

7.3 Coloring of intuitionistic fuzzy directed hypergraphs

Throughout this section, H refers to an IFDHG $H = (V, E)$.

Definition 7.3.1. Let H be an IFDHG. A *primitive p-coloring* A of H is a partition $\{A_1, A_2, A_3, ...A_p\}$ of V into p-subsets (colors) such that the support of each intuitionistic fuzzy hyperedge of H intersects atleast two colors of A, except spike edges.

Definition 7.3.2. Let H be an IFDHG. Let $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$. An $\mathcal{K}$-*coloring* A of H is a partition $\{A_1, A_2, A_3, ...A_p\}$ of V into p-subsets (colors) such that A induces a coloring for each core hypergraph H^{r_i,s_i} of H with $H^{r_i,s_i} = (V_i, E_i)$ where $V_i \subset V$ and $E_i \subset E$. The restriction of A to V_i, $\{A_1 \cap V_i, A_2 \cap V_i, A_3 \cap V_i, ...A_k \cap V_i\}$, is coloring of $\{H^{r_i,s_i}\}$. (Allow color set A_i to be empty).

Example 7.3.1. Consider an IFDHG, H with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{E_1, E_2, E_3, E_4\}$ whose adjacency matrix as follows:

$$H = \begin{array}{c c} & \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{pmatrix} \langle 0.8, 0\rangle & \langle 0.8, 0\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.8, 0\rangle & \langle 0.8, 0\rangle & \langle 0.8, 0\rangle & \langle 0, 1\rangle \\ \langle 0.7, 0.1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.7, 0.1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.6, 0.3\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.3, 0.2\rangle & \langle 0.3, 0.2\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix} \end{array}$$

The IF core hypergraphs of H are as follows:

$H^{0.8,0} = \{\{v_1, v_2\}, \{v_2\}\}$

$H^{0.7,0.1} = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_2\}, \{v_3\}\}$

$H^{0.6,0.3} = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_2, v_4\}, \{v_3, v_4\}\}$

$H^{0.3,0.2} = \{\{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_4\}\}$

The corresponding graph is shown in Figure 7.1.

Suppose $A = \{\{v_1, v_2\}, \{v_4\}, \{v_3, v_5\}\}$

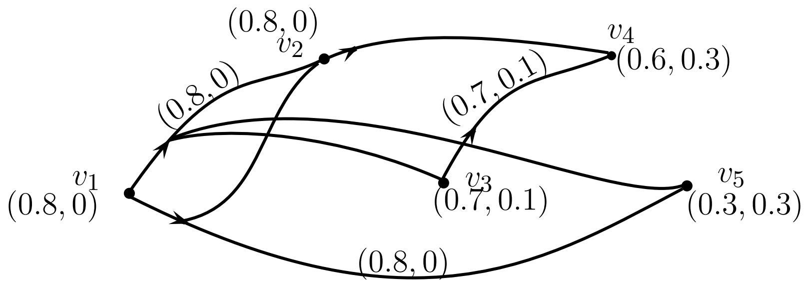


Figure 7.1: Intuitionistic fuzzy directed hypergraph

Then A is a coloring of $H^{0.6,0.3}$ and $H^{0.3,0.2}$ but not $H^{0.8,0}$. Hence A is a $\mathcal{K}-$coloring of H with intensity $\langle 0.8, 0\rangle$

Definition 7.3.3. The *p-chromatic number* of an IFDHG H is the minimal number $\chi_p(H)$, of colors needed to produce a primitive coloring of H. The *chromatic number* of H is the minimal number, $\chi(H)$, of colors needed to produce a $\mathcal{K}$-coloring of H.

Example 7.3.2. Consider an IFDHG, H where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ with adjacency matrix as below:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{pmatrix} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 \\ \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.6,0.3\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.5,0.2\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.2\rangle \\ \langle 0,1\rangle & \langle 0.3,0.1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.1\rangle & \langle 0,1\rangle & \langle 0.3,0.1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.2,0.1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.2,0.1\rangle & \langle 0.2,0.1\rangle \\ \langle 0,1\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0,1\rangle & \langle 0.5,0.2\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}$$

The corresponding graph is shown in Figure 7.2.

Then $C(H) = \{H^{r_i,s_i} = (V^{r_i,s_i}, E^{r_i,s_i}) | i = 1, 2, 3, 4\}$ where

$\langle r_1, s_1\rangle = \langle 0.6, 0.3\rangle; \langle r_2, s_2\rangle = \langle 0.5, 0.2\rangle; \langle r_3, s_3\rangle = \langle 0.3, 0.1\rangle; \langle r_4, s_4\rangle = \langle 0.2, 0.1\rangle$

$E_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$

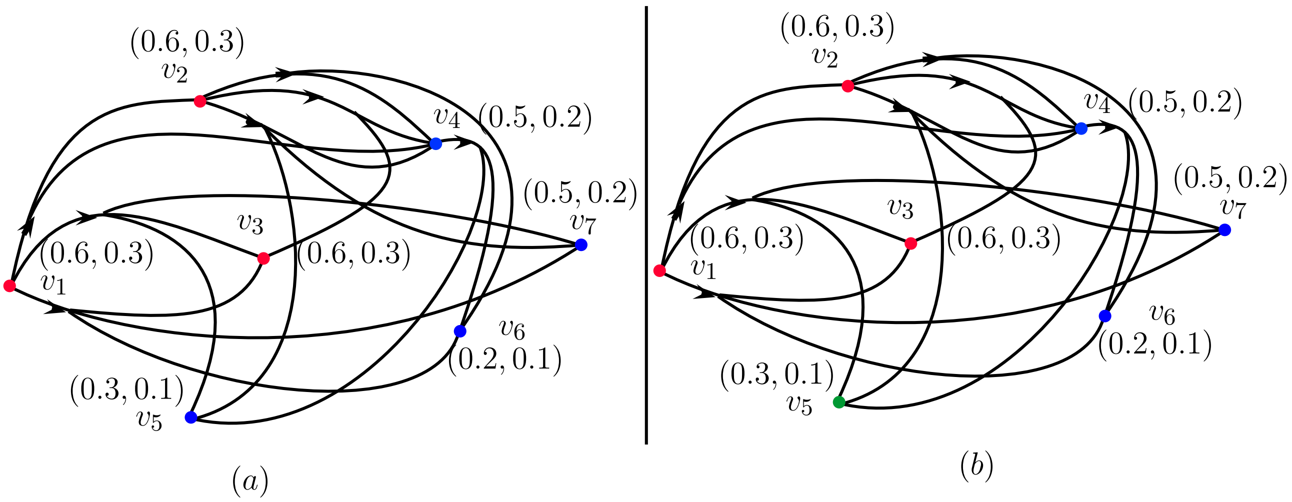


Figure 7.2: Intuitionistic fuzzy directed hypergraph with $(a)\chi(H^{r_2,s_2}) = 2$ and $(b)\chi(H^{r_3,s_3}) = 3$

$E_2 = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_7\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_7\}\}$

$E_3 = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_5, v_7\}, \{v_1, v_3, v_7\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_5\}, \{v_2, v_4\}, \{v_4, v_5\}\}$

$E_4 = \{\{v_1, v_2, v_4\}, \{v_1, v_3, v_5, v_7\}, \{v_1, v_3, v_6, v_7\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_5, v_7\},$

$\{v_2, v_4, v_6\}, \{v_4, v_5, v_6\}\}$

Consider H^{r_1,s_1}. Suppose $\{A_1, A_2\}$ is a coloring of H^{r_1,s_1}. Then $\{v_1, v_2\} \cap A_i \neq \emptyset, \{v_1, v_3\} \cap A_i \neq \emptyset, \{v_2, v_3\} \cap A_i \neq \emptyset$ for $i = 1, 2$.

Hence $A_1 \cap A_2 \neq \emptyset$, a contradiction. Thus $\chi(H^{r_1,s_1}) = 3$.

$\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6, v_7\}\}$ is a coloring for H^{r_2,s_2}, so $\chi(H^{r_2,s_2}) = 2$

For H^{r_3,s_3}, since $E \subseteq V, |E| = 3$. Hence $\chi(H^{r_3,s_3}) = 3$ and $\chi(H^{r_4,s_4}) = 3$.

Definition 7.3.4. A *spike reduction* of $E_i \in \mathcal{F}_\wp(V)$, denoted by $\widetilde{E}$, is defined as $\widetilde{E}(v_i) = max_i\{\langle r_i, s_i\rangle / \left|E_i^{r_i,s_i}\right| \geq 2, (0 \leq r_i \leq E_\mu(v_i), 0 \leq s_i \leq E_\nu(v_i))\}$.

Note: i) If $A = \phi$, then $\widetilde{E}(v_i) = 0$.

(ii) If E_i is spike, then $\widetilde{E} = \chi_0$

Definition 7.3.5. Let H be an IFDHG and let $\widetilde{H} = (\widetilde{V}, \widetilde{E})$, where $\widetilde{E} = \{\widetilde{E}_i | E_i \in E\}$ and $\widetilde{V} = \bigcup_{\widetilde{E}_i \in \widetilde{E}} supp(\widetilde{E})$.

Example 7.3.3. Consider example 7.3.2, $(E_7)^{0.5,0.2} = \{v_4\}$. Hence $\widetilde{E}_7(v_1) = \widetilde{E}_7(v_2) = \widetilde{E}_7(v_3) = 0$ and $\widetilde{E}_7(v_4) = \widetilde{E}_7(v_5) = \widetilde{E}_7(v_6) = \widetilde{E}_7(v_7) = (0.2, 0.1)$. It is clear that $\widetilde{E}_7 \neq E_7$. Since $\widetilde{E}_7 \neq \emptyset$, E_7 is not a spike.

Note: If each intuitionistic fuzzy hyperedge is spike, then $\widetilde{E} = \emptyset$. Hence $\widetilde{H}$ is not an IFDHG. Thus this concept cannot be proceeded in real coloring problem. So excluding it from further consideration and always proceed by assuming $\widetilde{H}$ exists.

Theorem 7.3.6. If H is an ordered IFDHG and A is a primitive coloring of H, then A is a $\mathcal{K}$-coloring of H.

Proof. Since H is an ordered IFDHG, from Definition 7.2.6, $C(H)$ is also an ordered IFDHG. That is, if $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$, then $H^{r_1,s_1} \subset H^{r_2,s_2} \subset ... \subset H^{r_n,s_n}$.

Since A is a primitive coloring of H, there exists a partition of V into p-subsets $\{A_1, A_2, A_3, ...A_p\}$ such that A induces a coloring for each core hypergraph, H^{r_i,s_i} of H. Hence A is a $\mathcal{K}$-coloring of H.

Theorem 7.3.7. Let H be an IFDHG and suppose $C(H) = \{H^{r_i,s_i} | i = 1, 2, 3...n\}$, where $0 \leq r_i \leq h_\mu(H)$ and $0 \leq s_i \leq h_\nu(H)$. If H^{r_n,s_n} is a simple IFDHG and singleton hyperedges do not appear in any core hypergraph of H and if each primitive coloring A of H is a $\mathcal{K}$-coloring of H, then H is an ordered IFDHG.

Proof. It is known that $H^{r_i,s_i} = (V^{r_i,s_i}, E^{r_i,s_i})$ for $1 \leq i \leq n$. Assume H^{r_n,s_n} is simple and that H is not ordered. Then there exists a primitive coloring of H that is not a $\mathcal{K}$-coloring of H.

Construction:

Since H is not ordered, there exists some core hypergraph H^{r_i,s_i}, where $i \leq n-1$, such that some hyperedges $E_i^{'} \in E_i$ is not an edge of $E_j, j > i$.

From definition 7.2.5, there is an intuitionistic fuzzy hyperedge $E_i \in E$ such that $E_i^{r_i,s_i} = E_i^{'}$. Let $\widetilde{E'}_i = E_i^{r_{i+1},s_{i+1}}$ and $F = E_i^{r_n,s_n}$. Then $E_i^{'} \subset \widetilde{E'}_i \subseteq F$.

Since H^{r_n,s_n} is simple and $F \in E_n$, it follows that $E_i^{'} \notin E_n$. Hence $|E_i^{'}| \geq 2$.

Hence, there is a primitive coloring of H that is not a $\mathcal{K}$-coloring of H.

Theorem 7.3.8. Let H be an ordered IFDHG and $C(H) = \{H^{r_i,s_i} | i = 1, 2, 3...n\}$, then $\chi(H^{r_1,s_1}) \leq \chi(H^{r_2,s_2}) \leq \cdots \leq \chi(H^{r_n,s_n}) = \chi(H)$, where $\chi(H^{r_i,s_i})$ represents the minimal number of colors required to color the crisp hypergraph H^{r_i,s_i}.

Definition 7.3.9. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be a pair of IFDHGs such that $V_1 \subseteq V_2$. Suppose $\mathcal{A}' = \{A_1, A_2, A_3, ...A_p\}$, where $\bigcup_{i=1}^{p} A_i = V_1$ and $A_i \neq \emptyset$, for $i = 1, 2, ...p$ is a $\mathcal{K}$- coloring or (p-coloring) of H_1. Then A'' is a *stable $\mathcal{K}$-coloring or (p-coloring)* extension of A' to H_2 if $A'' = \{A_1', A_2', A_3', ...A_p'\}$ is a $\mathcal{K}$-coloring or (p-coloring) of H_2 which satisfies

i) $\bigcup_{i=1}^{p} A_i' = V_2$

ii) $A_i \subseteq A_i'$ for $i = 1, 2, ...p$.

7.4 Skeleton of transversals of intuitionistic fuzzy directed hypergraph (H^s).

Let H be an IFDHG with fundamental sequence $F(H) = \{r_1, r_2, ..., r_n; s_1, s_2, ..., s_n\}$ where $0 \leq r_i \leq h_\mu(H)$ and $0 \leq s_i \leq h_\nu(H)$ and core set $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$.

Construction 1: The construction of $\widehat{C(H)}$ from $C(H)$ is a recursive process:

Step 1: Determine a IF partial hypergraph $\widehat{H}^{r_1,s_1}$ of H^{r_1,s_1} by eliminating all the IF hyperedge in H^{r_1,s_1} that properly contain another edge of H^{r_1,s_1}.

Step 2: Eliminate all IF hyperedges of H^{r_2,s_2} which are either properly contained another edge of H^{r_2,s_2} or contains (properly or improperly) an IF hyperedges of $\widehat{H}^{r_2,s_2}$. Then either all edges of H^{r_2,s_2} are eliminated or the remaining edges form an IFDHG $\widehat{H}^{r_2,s_2}$ of H^{r_2,s_2}.

Step i: For $i = 1, 2, 3, ...k$ where $1 \leq k \leq n-1$ and $n \geq 2$ this process is repeated.

Step k+1: Eliminate all IF hyperedges of $H^{r_{k+1},s_{k+1}}$ or contain an IF hyperedge of $\widehat{H}^{r_i,s_i}$ for $i = 1, 2, 3....k$ (if k exists). Then, either all edges of $H^{r_{k+1},s_{k+1}}$ are eliminated (and $\widehat{H}^{r_{k+1},s_{k+1}}$ does not exists) or the remaining IF hyperedges form a partial hypergraph $\widehat{H}^{r_{k+1},s_{k+1}}$ of $H^{r_{k+1},s_{k+1}}$. Continuing recursively upto n, we obtain $\widehat{F(H)} = \{r^s_1, r^s_2, ..., r^s_n; s^s_1, s^s_2, ..., s^s_n\}$ of $F(H)$. The IF coreset $C(H) = \{\widehat{H}^{r^s_1,s^s_1}, \widehat{H}^{r^s_2,s^s_2}, ..., \widehat{H}^{r^s_n,s^s_n}\}$ of IF partial hypergraph form $C(H)$.

Note: Each member of $\widehat{C(H)}$ has non-empty edge set and that for every $\langle r_i, s_i\rangle \in F(H)\backslash\{r^s_1, r^s_2,$
$..., r^s_n; s^s_1, s^s_2, ..., s^s_n\}$ the entire core hypergraph H^{r_i,s_i} was eliminated in the recursive process.

Definition 7.4.1. The *skeleton* of $\widetilde{H}$, denoted by $H^\Box$, is defined as $H^\Box = (\widetilde{H})^s$.

Theorem 7.4.2. Let H be an IFDHG and suppose for each H there exists a $\widetilde{H}$, then every p-coloring of $H^{\square}$ is a $\mathcal{K}$-coloring of $H^{\square}$ and conversely.

Proof. Since $H^{\square}$ is an ordered IFDHG, the result follows directly from Theorem 7.3.1.

Theorem 7.4.3. Let H be an IFDHG and there exists $\widetilde{H}$, then every $\mathcal{K}-$ coloring of H is a color stable extension of some p-coloring of $H^{\square}$. Conversely, any extended $\mathcal{K}$-coloring of $H^{\square}$ without adding any color is a color stable extended $\mathcal{K}$-coloring of H.

Example 7.4.1. Consider an IFDHG, H with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{E_1, E_2, E_3, E_4, E_5\}$ whose adjacency matrix is as given below:

$$H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \left(\begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \\ \langle 0.9, 0\rangle & \langle 0.9, 0\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.9, 0\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.7, 0.2\rangle & \langle 0.7, 0.2\rangle & \langle 0.7, 0.2\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.4, 0.2\rangle & \langle 0.4, 0.2\rangle & \langle 0.4, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0.3, 0.2\rangle & \langle 0, 1\rangle & \langle 0.3, 0.2\rangle & \langle 0.3, 0.2\rangle \end{array} \right)$$

The IF core hypergraphs are as follows:

$\langle r_1, s_1\rangle = \langle 0.9, 0\rangle; \langle r_2, s_2\rangle = \langle 0.7, 0.2\rangle; \langle r_3, s_3\rangle = \langle 0.4, 0.2\rangle; \langle r_4, s_4\rangle = \langle 0.3, 0.2\rangle$

$H^{0.9,0} = \{\{v_1\}\}$

$H^{0.7,0.2} = \{\{v_1, v_2\}, \{v_2\}\}$

$H^{0.4,0.2} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$

$H^{0.3,0.2} = \{\{v_1, v_2\}, \{v_1, v_2, v_4\}, \{v_2, v_3\}, \{v_2, v_3, v_4\}\{v_1, v_3, v_4\}\}$

No edge of H^{r_1,s_1} properly contains another edge of H^{r_i,s_i}. Hence $\widehat{H}^{r_i,s_i} = H^{r_i,s_i}$. For $H^{r_2,s_2}, \{v_1, v_2\} \supseteq \{v_1\}$. Thus, removing $\{v_1, v_2\}$ from E^{r_2,s_2} gives $\{v_2\}$. Hence $\widehat{H}^{r_2,s_2} = \{v_2\}$. For $H^{r_3,s_3}, \{v_2, v_3\} \supseteq \{v_2\}$. Removing edges which are properly contained in $\widehat{H}^{r_2,s_2}$ gives $\widehat{H}^{r_3,s_3} = \{v_1, v_3\}$.

It follows that $\langle {r_1}^s, {s_1}^s\rangle = \langle r_1, s_1\rangle$; $\langle {r_2}^s, {s_2}^s\rangle = \langle r_2, s_2\rangle$ and $\langle {r_3}^s, {s_3}^s\rangle = \langle r_3, s_3\rangle$.

Then $H^s = (V^s, E^s)$ where $V^s = \{v_1, v_2, v_3, v_4\}$ and $E^s = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$

Example 7.4.2. In Example 7.3.1, $\widetilde{H} = H$ and so $H^\square = H^s$. Every $\mathcal{K}$-coloring of H is a color stable extension of some $\mathcal{K}$-coloring of $H^\square$. But every $\mathcal{K}$-coloring of $H^\square$ is a $\mathcal{K}$-coloring of H. Since $E_1 = \{\{v_1\}\} = E_1^s$, $E_2^s = \{\{v_2\}\}$ and $E_2 = E_2^s \cup \{v_1, v_2\}$.

Example 7.4.3. Let H be an IFDHG. In Example 7.4.1, $H^s = (V^s, E^s)$ where $V^s = \{v_1, v_2, v_3, v_4, v_5\}$ and $E^s = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$. Hence $\{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$ are $\mathcal{K}$- coloring of H^s. Clearly chromatic number $\chi(H^s) = 2$. Note that $\{v_1, (0.9, 0)\}$ and $\{v_2, (0.7, 0.2)\}$ are spikes in H^s.

Consider spike reduction, $\widetilde{H} = (\widetilde{V}, \widetilde{E})$ where $\widetilde{V} = \{v_1, v_2, v_3, v_4\}$ and $\widetilde{E} = \{E_1, E_2, E_3, E_4, E_5\}$ which is represented by the adjacency matrix in Example 7.4.1:

Thus

$\widetilde{H}^{0.9,0} = \{\{v_1\}\}$

$\widetilde{H}^{0.7,0.2} = \{\{v_1, v_2\}, \{v_2\}\}$

$\widetilde{H}^{0.4,0.2} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$

$\widetilde{H}^{0.3,0.2} = \{\{v_1, v_2\}, \{v_1, v_2, v_4\}, \{v_2, v_3\}, \{v_2, v_3, v_4\}\{v_1, v_3, v_4\}\}$

Then $H^\square = \widetilde{H}^s = (V^\square, E^\square)$ where $V^\square = \{v_1, v_2, v_3, v_4, v_5\}$ and $E^\square = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$.

Hence $\{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$ are $\mathcal{K}$-coloring of $H^\square$. Clearly $\chi(H^\square) = 2$.

7.5 Chromatic values of intuitionistic fuzzy colorings

Definition 7.5.1. Let $H = (V, E)$ be an intuitionistic fuzzy directed hypergraph(IFDHG). The *lower truncation* H_l of H at (r_l, s_l)- level, $0 < r_l \leq h_\mu(H), 0 < s_l \leq h_\nu(H)$, where $r_l < \mu_i, s_l < \nu_i$, forall v_i, is an IFDHG, $H_l = \langle V_t, E_t, \mu_{t_l}(e_{ij}), \nu_{t_l}(e_{ij})\rangle$, where $V_t \subset V$ and $E_t \subset E$ denote the sets of vertices and edges of truncated IFDHG respectively and

$$\mu_{t_l}(e_{ij}) = \begin{cases} \mu_{ij} & \text{if} \quad \mu_{ij} \geq r_l \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{t_l}(e_{ij}) = \begin{cases} \nu_{ij} & \text{if} \quad \nu_{ij} \leq s_l \\ 1 & \text{otherwise} \end{cases}$$

are the membership and non-membership values of the edge e_{ij}.

The *upper truncation* H_u of H at (r_u, s_u)- level, $0 < r_u \leq h_\mu(H), 0 < s_u \leq h_\nu(H)$, where $r_u < \mu_i, s_u < \nu_i$, forall v_i, is an IFDHG, $H_u = \langle V_t, E_t, \mu_{t_u}(e_{ij}), \nu_{t_u}(e_{ij})\rangle$, where $V_t \subset V$ and $E_t \subset E$ denote the sets of vertices and edges of truncated IFDHG respectively and

$$\mu_{t_u}(e_{ij}) = \begin{cases} \mu_{ij} & \text{if} \quad \mu_{ij} \geq r_u \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{t_u}(e_{ij}) = \begin{cases} \nu_{ij} & \text{if} \quad \nu_{ij} \leq s_u \\ 1 & \text{otherwise} \end{cases}$$

are the membership and non-membership values of the edge e_{ij}.

Note: μ_{t_l}, μ_{t_u} are degrees of membership values of lower and upper truncation,

ν_{t_l}, ν_{t_u} are degrees of non-membership values of lower and upper truncation.

Example 7.5.1. Consider an IFDHG, H with the adjacency matrix as given below:

$$H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{pmatrix} \langle 0.7, 0.3\rangle & \langle 0, 1\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.6, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.5, 0.2\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.6, 0.2\rangle & \langle 0.6, 0.2\rangle \end{pmatrix} \end{array}$$

The corresponding graph of IFDHG H is displayed in Figure 7.3.

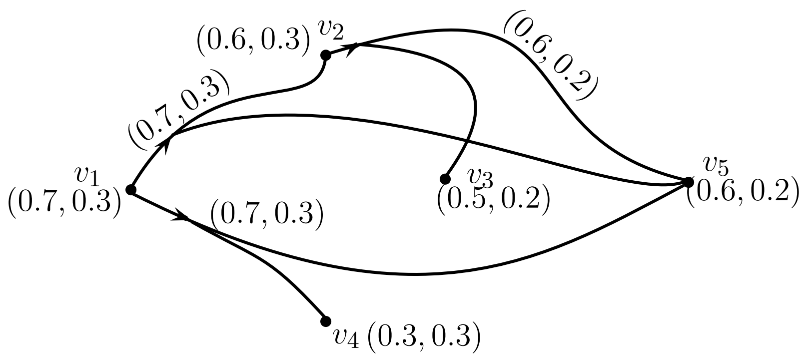


Figure 7.3: Intuitionistic fuzzy directed hypergraph H

The adjacency matrix of lower truncation of $H^{(0.6,0.3)}$ is given by

$$H = \begin{array}{c c} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{pmatrix} \langle 0.7, 0.3\rangle & \langle 0, 1\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.6, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix} \end{array}$$

The adjacency matrix of upper truncation of $H^{(0.6,0.3)}$ is given by

$$H = \begin{array}{c c} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{pmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.5, 0.2\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.3\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix} \end{array}$$

The graphs of lower and upper truncations are given in Figure 7.4:

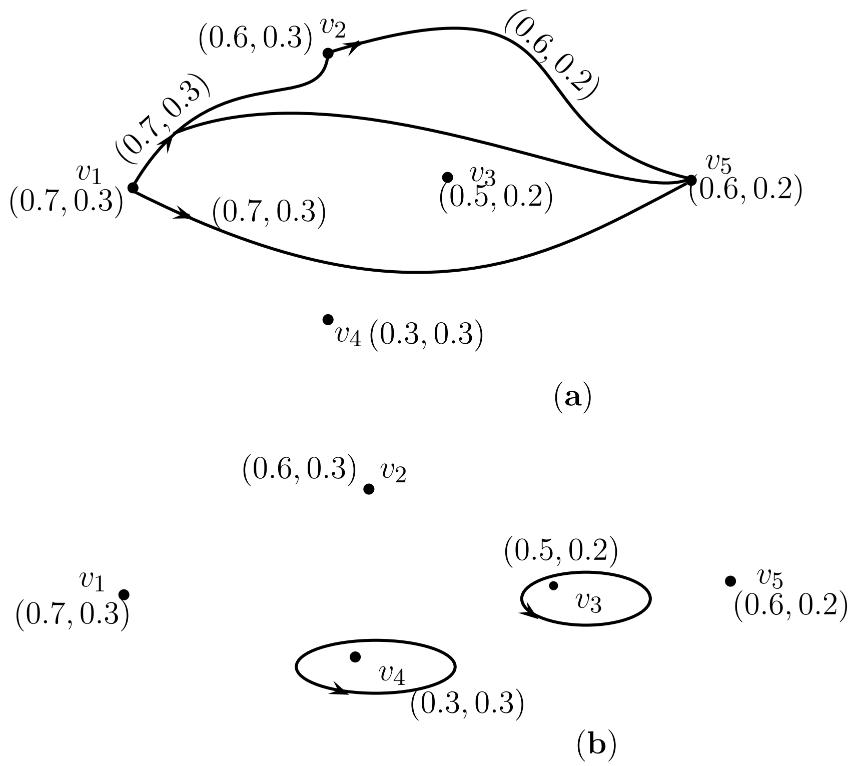


Figure 7.4: (**a**) Lower Truncation of $H^{(0.6,0.3)}$ and (**b**) Upper Truncation of $H^{(0.6,0.3)}$

.

Note:

1. $H_l \cup H_u \subseteq H$.

2. $H_l \cap H_u = \phi$.

Definition 7.5.2. Let H be an IFDHG with core set $C(H) = \{H^{r_i,s_i} = (V^{r_i,s_i}, E^{r_i,s_i})/i = 1, 2, ...n\}$ where $E(H^{r_i,s_i}) = E_i$ is the crisp edge set of the core hypergraph H^{r_i,s_i}. Let $E(H)$ denote the crisp edge set of H defined by $E(H) = \cup\{E_i/E_i = E(H^{r_i,s_i}); H^{r_i,s_i} \in C(H)\}$. $E(H)$, a crisp hypergraph on V, is called *core aggregate hypergraph* of H and is denoted by $\mathcal{H}(H) = (V, E(H))$.

Theorem 7.5.3. For every intuitionistic fuzzy hypergraph $\mathcal{H}$, a p-coloring of

$\mathcal{H}(H)$ is a $\mathcal{K}$-coloring of H and conversely.

Definition 7.5.4. Let H be an IFDHG. Then, every $\mathcal{K}-$ colorings of H which is a conservative p - coloring of $\mathcal{H}(H)$, is called a *conservative* $\mathcal{K}-$ *colorings of* H.

Definition 7.5.5. Let $H^{r_j,s_j} = (V^{r_j,s_j}, E^{r_j,s_j})$ be a (crisp) core directed hypergraph of an IFDHG H where $V^{r_i,s_i} = V \backslash V^{r_j,s_j} \neq \phi$. Suppose L is a $\mathcal{K}-$ colorings of upper truncated IFDHG, H^{r_j,s_j} which is obtained by extending a p-coloring, L_j of H^{r_j,s_j}. If L is weekly (or strongly) conservative p - coloring extension of L_j to the (crisp) core aggregate hypergraph $H(H^{r_j,s_j})$ of H^{r_j,s_j} with respect to V^{r_i,s_i}, then L is called a weekly (or strongly) conservative $\mathcal{K}$ - coloring extension of L_j with respect to V^{r_i,s_i}.

Definition 7.5.6. Let H be an IFDHG and suppose $\Lambda = \{\delta_i \in \mathcal{IF}_p(v)/i = 1, 2, ...p\}$ is a finite subset of $\mathcal{IF}_p(v)$. Then Λ is called *intuitionistic fuzzy coloring of* H if the following properties are satisfied:

1) $h(\Lambda) = \langle max(\mu_{ij}(v_i), min(\nu_{ij}(v_i))\rangle$, for all $v_i \in V$

2) $\delta_i \cap \delta_j = \phi$ if $i \neq j$

3) Λ^{r_i,s_i} is a coloring of H^{r_i,s_i} for $0 < r_i < h_\mu(H)$ and $0 < s_i < h_\nu(H)$.

Theorem 7.5.7. For every intuitionistic fuzzy hypergraph $\mathcal{H}$, a p-coloring of $\mathcal{H}(H)$ is a $\mathcal{K}$-coloring of H and vice-versa.

Note: Λ is sequentially elementary with respect to $F(H)$. There is one-to-one correspondance between the $\mathcal{K}-$ coloring of H and the intuitionistic fuzzy coloring of H, if the color set is empty.

Let Λ is an IFC of $H = (V, E)$. Then by Definition 7.5.1, (r_n, s_n)-cut, Λ^{r_n,s_n} of Λ, where (r_n, s_n) is the smallest value in $F(H)$, is p- coloring of the core aggregate hypergraph $\mathcal{H}(H)$ of H which implies Λ^{r_n,s_n} is a $\mathcal{K}-$ coloring of H by Theorem 7.5.1.

Conversely, suppose $\mathcal{A} = \{A_1, A_2, ...A_k\}$ is a $\mathcal{K}-$ coloring of $\mathcal{H}(H)$. Then $\mathcal{A}$ is a crisp coloring of the core aggregate hypergraph $\mathcal{H}(H)$ of $H, \cup_{i=1}^{k} A_k = V$ and $A_i \cap A_j = \phi$ if $i \neq j$. Now A_i, associate an intuitionistic fuzzy subset $\delta_i \in \mathcal{IF}_p(v)$, with support A_i, defined by

$$\delta_i(v_i) = \begin{cases} \langle max(\mu_{ij}(v_i)), min(\nu_{ij}(v_i))\rangle | \mu_{ij}, \nu_{ij} \in E & \text{if } v_i \in A_i \\ \langle 0, 1\rangle & \text{otherwise} \end{cases}$$

Hence $\Lambda = \{\delta_1, \delta_2, ..., \delta_k\}$ is an IFC of H.

Definition 7.5.8. Let $\delta_i \in \mathcal{IF}_p(v)$. Then the intuitionistic fuzzy subset $\delta_{i(c)}$ of V for all $v_i \in V$ is defined by

$$\delta_{i(c)}(v) = \begin{cases} h(\delta) & \text{if } \delta_i(v_i) = \langle max(\mu_{ij}(v_i)), min(\nu_{ij}(v_i))\rangle | \mu_{ij}, \nu_{ij} \in E \\ \langle 0, 1\rangle & \text{otherwise} \end{cases}$$

$\delta_{i(c)}$ is called *the elementary center.*

Definition 7.5.9. Let $\Lambda = \{\delta_i \in \mathcal{IF}_p(v)/i = 1, 2, ...p\}$. Then $\Lambda_{i(c)}$ is called *elementary center of* Λ, is defined by $\Lambda_{i(c)} = \{\delta_{1(c)}, \delta_{2(c)}, ...\delta_{p(c)}\}$, where $\delta_{i(c)}$ is the elementary center of δ_i.

Definition 7.5.10. Let $\Lambda_{(c)}$ be the elementary center of IFC Λ of H with fundamental sequence $F(\Lambda_{(c)}) = \{u_1^\Lambda, u_2^\Lambda, ..., u_m^\Lambda\}$, where $u_1^\Lambda > u_2^\Lambda > ...u_m^\Lambda$ and let t be a monotonic increasing function on the interval [0, 1] such that $t(0) = 0$ and $t(1) = 1$. Such t is called *scaling function.*

Definition 7.5.11. Let H be an IFDHG and let t denote a scaling function. Then $\chi_t(H) = min\{\Gamma_t(\Lambda)/\Lambda$ is an IFC of $H\}$ and $\hat{\chi}_t(H) = min\{\hat{\Gamma}_t(\Lambda)/\Lambda$ is an IFC of $H\}$ are called Γ_t - *chromatic number* and $\hat{\Gamma}_t$ - *chromatic number* of H respectively.

Note: If t is the identity mapping on [0, 1], then Γ_t or $\hat{\Gamma}_t$ are called linear chromatic numbers of H.

Theorem 7.5.12. Let H be an IFDHG then for every H and for every scaling function $t : [0, 1] \to [0, 1], \chi_t(H) \leq \chi(H), \hat{\chi}_t(H) \leq \hat{\chi}(H)$ and $\chi(H) = min\{|\Lambda|/\Lambda$ is an IFC of $H\} = min\{|L|/L$ is a $\mathcal{K}$ - coloring of $H\}$ where $|\Lambda|$ is the number of edges in Λ and $|L|$ is the number of colors in L.

Example 7.5.2. Consider an IFDHG, H with $V = \{v_1, v_2, v_3, \cdots, v_9\}$ and $E = \{E_1, E_2, \cdots, E_{15}\}$:

Here $V = \{v_1, v_2, \cdots, v_9\}$, $E = \{E_1, E_2, \cdots, E_{15}\}$ and $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}\}$ where $H^{r_1,s_1} = (\{v_1, v_2, \cdots, v_6\}, \{E_1, E_2, \cdots, E_6\})$ and $H^{r_2,s_2} = (V, E)$.
Since H is elementary, it is ordered.
Thus every primitive coloring of H is an $\mathcal{K}-$ coloring of H. Therefore $\chi(H) = 3$, since H^{r_2,s_2} has the following primitive coloring: $A_1 = \{Blue(B), Green(G), Yellow(Y)\}$ where $B = \{v_1, v_4, v_9\}$, $G = \{v_2, v_6, v_8\}$ and $Y = \{v_3, v_5, v_7\}$.
Suppose t is the identity map. Assume $\chi_t(H) = \Gamma_t(\Lambda)$. It is interesting to compare $\Gamma_t(\Lambda_1)$ with $\Gamma_t(\Lambda_2)$, where Λ_1 and Λ_2 are the IFC of H.

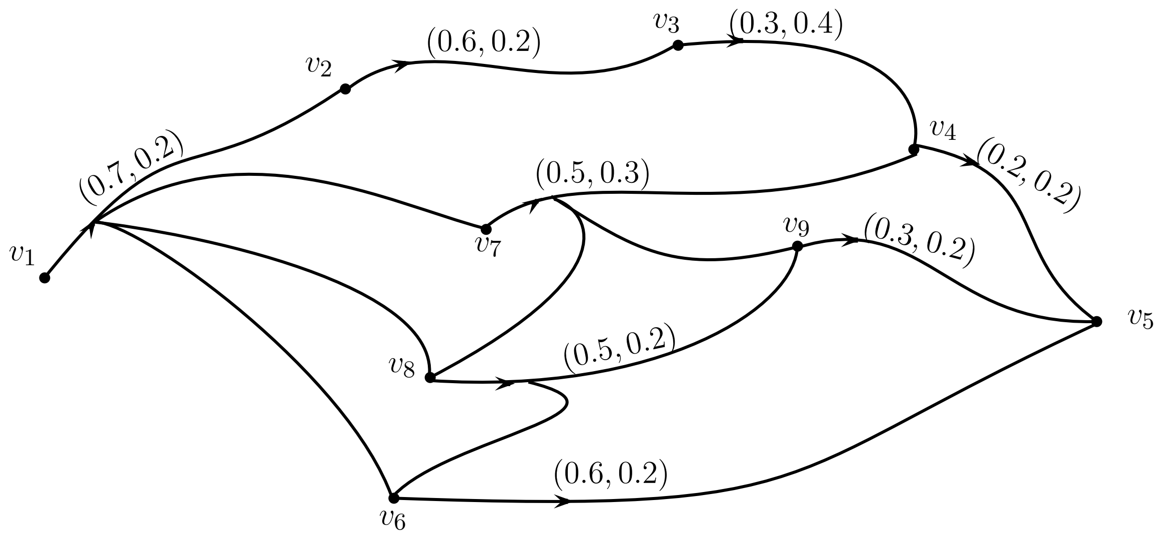


Figure 7.5: Chromatic Numbers of H

.

Let $A_2 = \{Blue(B), Green(G), Yellow(Y), Red(R), White(W)\}$, where $B = \{v_1, v_3, v_5\}$, $G = \{v_7\}$, $R = \{v_8\}$,$W = \{v_9\}$ and $Y = \{v_2, v_4, v_6\}$. The restriction, A_2' of A_2 to H^{r_1,s_1} is $\chi(H) = 2$, $A_2' = \{B, Y\}$.

7.6 Intersecting intuitionistic fuzzy directed hypergraph

Let $H = (V, E)$ be an intuitionistic fuzzy directed hypergraph (IFDHG).

Definition 7.6.1. An IFDHG $H = (V, E)$ is said to be *intersecting intuitionistic fuzzy directed hypergraph*, if for each pair of intuitionistic fuzzy hyperedge $\{E_i, E_j\} \subseteq E$, $E_i \bigcap E_j \neq \phi$.

Definition 7.6.2. Let $H = (V, E)$ be an IFDHG and $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$, if H^{r_i,s_i} is an intersecting intuitionistic fuzzy directed hypergraph for each $i = 1, 2, ..., n$ then H is $\mathcal{K}$-*intersecting IFDHG*.

Definition 7.6.3. An IFDHG $H = (V, E)$ is said to be *strongly intersecting*, if for any two edges E_i and E_j contain a common spike of height, $h = h(E_i) \wedge h(E_j)$.

Theorem 7.6.4. Let $H = (V, E)$ be an IFDHG and suppose $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$. Then H is intersecting if and only if $H^{r_n,s_n} = (V^{r_n,s_n}, E^{r_n,s_n})$ is intersecting.

Proof:

H is intersecting $\Longleftrightarrow$supp(H)= $\{supp(E_j)/E_j \in E\}$ is intersecting, from Definition 7.5.11

Similarly, each pair of intuitionistic fuzzy hyperedges, $\{E_1, E_2, ..., E_n\} \subseteq E$

$H^{r_1,s_1}, H^{r_2,s_2}, \cdots, H^{r_n,s_n}$ are intersecting.

Conversely, let $H^{r_n,s_n} = (V^{r_n,s_n}, E^{r_n,s_n})$ is intersecting

Since, $supp(H) = \{supp(E_j)/E_j \in E\}$ is intersecting, H is also intersecting.

Theorem 7.6.5. Let $H = (V, E)$ be an ordered intuitionistic fuzzy directed hypergraph and let $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$, then H is intersecting if and only if H is $\mathcal{K}$-intersecting.

Proof:

The proof is direct from Definition 7.6.1 and Theorem 7.5.12.

Theorem 7.6.6. Suppose $H = (V, E)$ is an ordered intersecting IFDHG, then each intuitionistic fuzzy hyperedge T of H contains a member of $Tr(H_{h(T)})$, where $H_{h(T)}$ is the upper truncation of H at level $h(T)$. In particular T is an intuitionistic fuzzy transversal of $H_{h(T)}$.

Proof:

Let $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$ and suppose $T_j \in E$. Assume that,

$(r_1, s_1) = h(T)$, since H is ordered and $T^{r_1,s_1} \neq \phi$.

Since H is intersecting $\Rightarrow H^{r_n,s_n}$ is also intersecting.

Therefore, T^{r_1,s_1} is an intuitionistic fuzzy transversal of H^{r_n,s_n}. Let T_1 be a minimal intuitionistic fuzzy transversal of H^{r_n,s_n} contained in T^{r_1,s_1}. Since H is ordered, then there is a nested sequence of sets

$$T_n \supseteq ....T_i \supseteq .... \supseteq T_1$$

such that, T_i is a minimal intuitionistic fuzzy transversal of H^{r_i,s_i} for every $(r_i, s_i) \in F(H)$

Let θ_i be the elementary intuitionistic fuzzy subset with support T' and height (r_i, s_i), for i=1,2,...,n. Then clearly,

$$\bigcup_{i=1}^{n} \theta_i \in Tr(H) \ and \ T' \subseteq T.$$

Therefore, each intuitionistic fuzzy hyperedge T of H contains a member of $Tr(H_{h(T)})$.

Theorem 7.6.7. If $H = (V, E)$ is a simple, intersecting IFDHG such that $\chi(H) > 2$, then $E = \{T' | T' \in min(Tr(H))\}$

Corollary 7.6.8. Suppose if theorem 7.6.7 holds good for $\chi(H) > 2$, then H has no loops.

Theorem 7.6.9. Let H be an ordered, intersecting IFDHG with $C(H) = \{H^{r_1,s_1}, H^{r_2,s_2}, ..., H^{r_n,s_n}\}$. Suppose that $\chi(H^{r_1,s_1}) > 2$ and H^{r_n,s_n} is simple. Then for each $\langle r_i, s_i \rangle \in F(H)$,

$$Tr(H^{r_i,s_i}) = \{\theta(T, \langle r_i, s_i \rangle / T \in H^{r_i,s_i}\}$$

where $\theta(T, \langle r_i, s_i \rangle)$ is an elementary intuitionistic fuzzy subset with support E and height $\langle r_i, s_i \rangle$.

Proof:

By hypothesis, it follows that H^{r_i,s_i} is simple, intersecting and $\chi(H^{r_i,s_i}) > 2$ for each $H^{r_i,s_i} \in C(H)$.

By theorem 7.6.7, T is the set of all minimal transversals of H. Thus the set of $H^{r_i,s_i} = Tr(H^{r_i,s_i})$, for every $(r_i, s_i) \in F(H)$. Hence the desired result.

Theorem 7.6.10. Let H be an IFDHG. Then H is strongly intersecting if and only if H is $\mathcal{K}$-intersecting.

Proof:

Necessary Part: Suppose that H is strongly intersecting, let E_i and E_j be edges of $H^{r_i,s_i} \in C(H)$. Then there exists two edges E_1 and E_2 of H such that, $E_1^{r_i,s_i} = E_1$ and $E_2^{r_i,s_i} = E_2$.

Since H is strongly intersecting, both E_1 and E_2 contain a common spike θ_{v_i}, where $0 \leq r_i \leq h_\mu(\theta_{v_i})$ and $0 \leq s_i \leq h_\nu(\theta_{v_i})$

Thus, $supp(\theta_{v_i}) = \{v_i\} \subseteq E_i \cap E_j$. Hence H^{r_i,s_i} is intersecting and H is $\mathcal{K}$-intersecting.

Sufficient Part: Suppose that H is $\mathcal{K}$-intersecting, let F_i and F_j be hyperedges of H and let $\langle r_i, s_i \rangle = h(F_i) \wedge h(F_j)$ and let $E_i = F_i^{r_i,s_i}$, $E_j = F_j^{r_i,s_i}$, then both

$E_i, E_j \in H^{r_i,s_i} = H^{r_j,s_j}$, where $r_{j+1} < r_i \leq r_j, s_{j+1} < s_i \leq s_j$.

Let $\langle r_{n+1}, s_{n+1}\rangle = \langle 0, 1\rangle$, since H^{r_i,s_i} is intersecting, there exists a vertex $v_i \in E_i \cap E_j$. There is a spike θ_{v_i} with support $\{v_i\}$ and height $\langle r_i, s_i\rangle$ which is contained in both F_i and F_j.

Hence H is strongly intersecting.

Theorem 7.6.11. If H^s is intersecting, then H is strongly intersecting.

Proof:

Let $C(H) = \{\mathcal{H}^{r_i,s_i} = (V_i, E_i) | i = 1, 2, ..., n\}$ be the set of core intuitionistic fuzzy hypergraphs of H and consider the core's aggregate intuitionistic fuzzy hypergraph,

$\mathcal{H}(H) = (V, E(H))$, where $E(H) = \cup \{E_i | i = 1, 2, ..., n\}$

In addition, let $(\mathcal{H}^s)^{r^s_m, s^s_m} = (V^s_m, E^s_m)$, represent the core hypergraph of H^s associated with the smallest member (r^s_m, s^s_m) of $F(H)$.

From the construction of H^s it follows that every edge belonging to $E(H)$ contains an edge of E^s_m.

Hence $\mathcal{H}^s$ is intersecting $\Rightarrow H$ is strongly intersecting.

If H^s is intersecting, then by Theorem 7.6.7, $(\mathcal{H}^s)^{r^s_m}$ is intersecting, and therefore, the family of (crisp) edges $E(H)$ is intersecting as well.

Example 7.6.1. Consider an IFDHG H with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{E_1, E_2, E_3\}$ whose incidence matrix is as follows:

$$H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.7, 0.2\rangle & \langle 0.7, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0.5, 0.2\rangle & \langle 0.5, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.3, 0.2\rangle & \langle 0, 1\rangle & \langle 0.3, 0.2\rangle \end{pmatrix} \end{array}$$

Clearly, $h(H) = \langle 0.7, 0.2\rangle$.

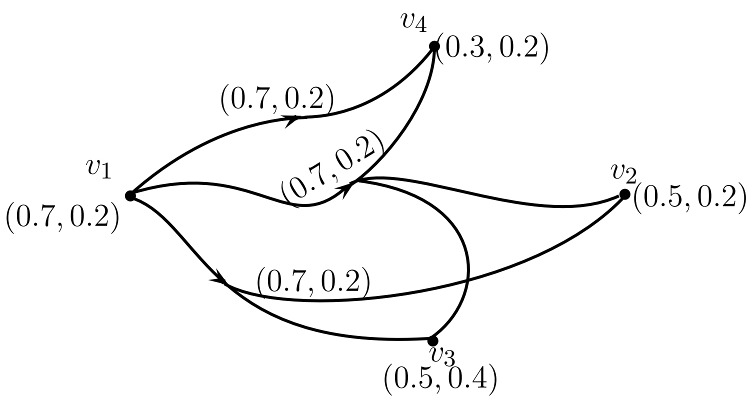


Figure 7.6: $\mathcal{K}-$ intersecting IFDHG

.

Then,

$E^{0.7,0.2} = \{\{v_1, v_4\}\}$

$E^{0.5,0.2} = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}\}$

$E^{0.5,0.4} = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}\}$

$E^{0.3,0.2} = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4\}\}$

Thus, $0.3 < r \leq 0.7$ and $0.2 \leq s \leq 0.4$

$E^{r,s} = \{v_1\} = E^{0.7,0.2}$

and for $0 < r \leq 0.3$ and $0.4 \leq s < 1$

$$E^{r,s} = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4\}\} = E^{0.3,0.2}$$

Note that, $E^{0.7,0.2} \subset E^{0.3,0.2}$

Therefore, $E^{0.7,0.2} \subset E^{0.5,0.2} \subset E^{0.5,0.40.3,0.2}$

Thus, H is an ordered intuitionistic fuzzy directed hypergraph.

$$H^{0.7,0.2} = (V_1, E_1) = \{\{v_1, v_4\}\}$$

$$H^{0.5,0.2} = (V_2, E_2) = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}\}$$

$$H^{0.5,0.4} = (V_3, E_3) = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}\}$$

$$H^{0.3,0.2} = (V_4, E_4) = \{\{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4\}\}$$

Thus H is a $\mathcal{K}$-intersecting intuitionistic fuzzy directed hypergraph.

Example 7.6.2. Consider an IFDHG $H = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{E_1, E_2, E_3\}$ which is represented by the following incidence matrix:

$$H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{pmatrix} \langle 0.6, 0.4\rangle & \langle 0, 1\rangle & \langle 0.6, 0.4\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0.2\rangle & \langle 0.5, 0.2\rangle & \langle 0.5, 0.2\rangle \end{pmatrix} \end{array}$$

Clearly, the *height* of an intuitionistic fuzzy set $h(E_j) = \langle max(\mu_{ij}), min(\nu_{ij})\rangle$, hence $h(H) = (0.6, 0.4)$

$$E^{0.6,0.4} = \{\{v_1\}\}$$

$$E^{0.5,0.2} = \{\{v_1, v_3\}\}$$

$$E^{0.4,0.3} = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_1, v_3\}\}$$

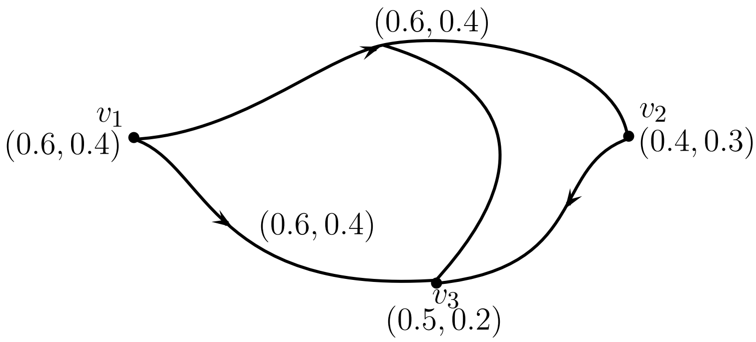


Figure 7.7: Not - ordered intersecting IFDHG

.

Therefore, $E^{0.6,0.4} \sqsubset E^{0.5,0.2} \sqsubset E^{0.4,0.3}$ but, $E^{0.6,0.4} \not\subseteq E^{0.5,0.2}$. Hence, H is not ordered.

$H^{0.6,0.4} = (V_1, E_1) = (\{v_1\}, \{\{v_1\}\})$

$H^{0.5,0.2} = (V_2, E_2) = (\{v_1, v_3\}, \{\{v_3\}, \{v_1, v_3\}\})$

$H^{0.4,0.3} = (V_3, E_3) = (\{v_1, v_2, v_3\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_1\}\})$

Thus H is not a $\mathcal{K}$-intersecting IFDHG.

Chapter 8

An application of intuitionistic fuzzy directed hypergraph in molecular structure representation

8.1 Introduction

The first definition of fuzzy graphs was proposed by Kaufmann, from the fuzzy relations introduced by Zadeh. Although Rosenfeld introduced another elaborated definition, including fuzzy vertex and fuzzy edges, the first definition of intuitionistic fuzzy graphs was proposed by Atanassov[9]. He defined intuitionistic fuzzy graph as the set

$G = \{\langle\langle x, y\rangle, \mu_G(x, y), \nu_G(x, y)\rangle | \langle x, y\rangle \in E_1 \times E_2\}$ if the functions $\mu_G : E_1 \times E_2 \to [0, 1]$ and $\nu_G : E_1 \times E_2 \to [0, 1]$ define the degree of membership and the degree of non-membership, respectively, of the element $\langle x, y\rangle \in E_1 \times E_2$ to the set $G \subset E_1 \times E_2$ and for all $\langle x, y\rangle \in E_1 \times E_2 :\ 0 \leq \mu_G(x, y) + \nu_G(x, y) \leq 1$. An *intuitionistic fuzzy hypergraph* (IFHG) [79] is an ordered pair $H = (V, \mathcal{E})$ where

(i) $V = \{v_1, v_2, ..., v_n\}$, is a finite set of intuitionistic fuzzy vertices,

(ii) $\mathcal{E} = \{E_1, E_2, ..., E_m\}$ is a family of crisp subsets of V,

(iii) $E_j = \{(v_i, \mu_j(v_i), \nu_j(v_j)) : \mu_j(v_i), \nu_j(v_i) \geq 0$ and $\mu_j(v_i) + \nu_j(v_i) \leq 1\}, j = 1, 2, ..., m$,

(iv) $E_j \neq \phi, j = 1, 2, ..., m$,

(v) $\bigcup_j supp(E_j) = V, j = 1, 2, ..., m$.

Here, the hyperedges E_j are crisp sets of intuitionistic fuzzy vertices, $\mu_j(v_i)$ and $\nu_j(v_i)$ denote the degrees of membership and non-membership of vertex v_i to edge E_j. Thus, the elements of the incidence matrix of IFHG are of the form $(v_{ij}, \mu_j(v_i), \nu_j(v_j))$. The sets $(V, \mathcal{E})$ are crisp sets. In [68], the intersecting IFDHG, $\mathcal{K}$-intersecting IFDHG and strongly intersecting IFDHG were studied. Here, some more intersecting concepts of fuzzy hypergraphs in [65] are extended to IFDHGs. This chapter has five sections: Section 2 gives the notations and theorem which are used in this work. In section 3, essentially intersecting, essentially strongly intersecting, skeleton intersecting, non-trivial, sequentially simple and essentially sequentially simple IFDHGs are defined. Also, it has been proved that if IFDHG H is ordered and essentially intersecting, then $\chi(H) \leq 3$. An IFDHG H is strongly intersecting if and only if $H^{\langle r_i, s_i \rangle}$ is intersecting for every $\langle r_i, s_i \rangle \in F(H)$ is proven and an application of IFDHG in molecular structure representation is also given.

8.2 Notations and Prerequisites

The notations and theorem used in this work are given below:

$H = (V, \mathcal{E})$	- IFDHG with vertex set V and edge set $\mathcal{E}$
$\langle \mu_i, \nu_i \rangle$	- degrees of membership and non-membership of the vertex v_i
$\langle \mu_{ij}, \nu_{ij} \rangle$	- degrees of membership and non-membership of the i^{th} vertex in j^{th} edge
$\langle \mu_{ij}(v_i), \nu_{ij}(v_i) \rangle$	- degrees of membership and non-membership of the edges containing v_i
$h(H)$	- height of a hypergraph H
$F(H)$	- Fundamental sequence of H
$Tr(H)$	- Intuitionistic Fuzzy Transversals (IFT) of H
$C(H)$	- Core set of H
$H^{\langle r_i, s_i \rangle}$	- $\langle r_i, s_i \rangle$ - level of H
$\mathcal{IF}_p(V)$	- Intuitionistic Fuzzy power set of V.
$\widetilde{E}_j$	- spike reduction of $E_j \in \mathcal{IF}_p(V)$
ϕ	- empty IFS (i.e., IFS having elements with zero membership and unit nonmembership values)

Theorem 8.2.1. [68] Let H be an IFDHG. Then H is strongly intersecting if and only if H is $\mathcal{K}$-intersecting.

8.3 Intersecting Intuitionistic Fuzzy Directed Hypergraphs

In this section, essentially intersecting, essentially strongly intersecting, skeleton intersecting, non-trivial, sequentially simple and essentially sequentially simple IFDHGs are defined.

8.3.1 Essentially Intersecting IFDHGs

Definition 8.3.1. A spike reduction of $E_j \in \mathcal{IF}_p(V)$, denoted by $\widetilde{E_j}$, is defined by

$$\widetilde{E_j}^{\langle r_j,s_j\rangle} = \begin{cases} E_j{}^{\langle r_j,s_j\rangle} & \text{if } |E_j{}^{\langle r_j,s_j\rangle}| \geq 2 \\ \phi & \text{if } |E_j{}^{\langle r_j,s_j\rangle}| < 2 \end{cases}$$

where r_j=min $\{\mu_j(v_i)\} \in (0,1]$ and s_j=max $\{\nu_j(v_i)\} \in [0,1)$

Definition 8.3.2. Let $H = (V, \mathcal{E})$ be an IFDHG. The spike reduced IFDHG of H, denoted by $\widetilde{H}$, is defined as $\widetilde{H} = (\widetilde{V}, \widetilde{\mathcal{E}})$, where $\widetilde{\mathcal{E}} = \{\widetilde{E_j} | E_j \in \mathcal{E}\}$; $\widetilde{V} = \bigcup_{\widetilde{E_j}\in\widetilde{\mathcal{E}}} supp(\widetilde{\mathcal{E}})$ and

$$\langle \mu_j(\widetilde{v_i}), \nu_j(\widetilde{v_i})\rangle = \begin{cases} \langle r_j, s_j\rangle & \text{if } \widetilde{v_i} \in supp(\widetilde{E}_j) \\ \langle 0,1\rangle & \text{if } \widetilde{v_i} \notin supp(\widetilde{E}_j) \end{cases}$$

Example 8.3.1. Consider an IFDHG H with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$ whose incidence matrix as follows:

$$H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{pmatrix} \langle 0.8, 0.1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.5, 0.2\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.8, 0.1\rangle & \langle 0.3, 0.4\rangle & \langle 0,1\rangle \\ \langle 0.3, 0.6\rangle & \langle 0,1\rangle & \langle 0.6, 0.2\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.5, 0.1\rangle \end{pmatrix} \end{array}$$

Then the incidence matrix of $\widetilde{H} = (\widetilde{V}, \widetilde{\mathcal{E}})$, where $\widetilde{\mathcal{E}} = \{\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\}$ and $\widetilde{V} = \{\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3, \widetilde{v}_4, \widetilde{v}_5\}$ is as follows:

$$\widetilde{H} = \begin{array}{c} \\ \widetilde{v}_1 \\ \widetilde{v}_2 \\ \widetilde{v}_3 \\ \widetilde{v}_4 \\ \widetilde{v}_5 \end{array} \begin{array}{c} \begin{array}{ccc} \widetilde{E}_1 & \widetilde{E}_2 & \widetilde{E}_3 \end{array} \\ \begin{pmatrix} \langle 0.3, 0.6\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.3, 0.6\rangle & \langle 0.5, 0.2\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.5, 0.2\rangle & \langle 0.3, 0.4\rangle \\ \langle 0.3, 0.6\rangle & \langle 0, 1\rangle & \langle 0.3, 0.4\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix} \end{array}$$

Note: It is to be noted that there are two changes happened in $\widetilde{H}$:

(i) The spike is reduced;

(ii) The degrees of membership and nonmembership of the vertices have been modified. The graphs of H and $\widetilde{H}$ are given in Figure 8.1.

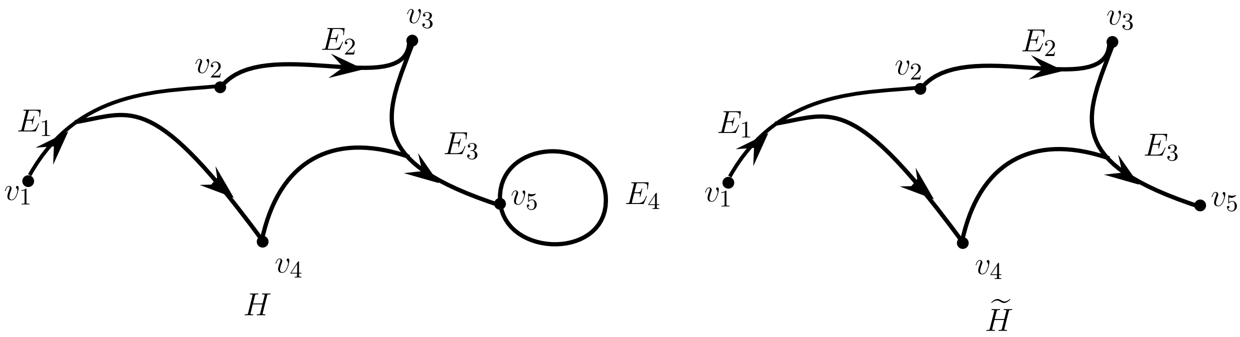


Figure 8.1

Definition 8.3.3. A IFDHG H is said to be *essentially intersecting* if $\widetilde{H}$ is intersecting. H is said to be *essentially strongly intersecting* if $\widetilde{H}$ is strongly intersecting.

Theorem 8.3.4. If IFDHG H is ordered and essentially intersecting, then $\chi(H) \leq 3$.

Proof: Assume that $\widetilde{H}$ exists, for otherwise $\chi(H)$=1. Let $(\widetilde{H})^{\langle r_m,s_m\rangle} \in C(\widetilde{H})$, where $\langle r_m, s_m\rangle$ is the smallest value of $F(\widetilde{H})$. Since $\widetilde{H}$ is intersecting, it follows from known theorem "Let H be an IFDHG and suppose $C(H) = \left\{H^{\langle r_1,s_1\rangle}, H^{\langle r_2,s_2\rangle}, ..., H^{\langle r_n,s_n\rangle}\right\}$, then H is intersecting if and only if $H^{\langle r_n,s_n\rangle} = (V^{\langle r_n,s_n\rangle}, \mathcal{E}^{\langle r_n,s_n\rangle})$ is intersecting." that $(\widetilde{H})^{\langle r_m,s_m\rangle}$ is a crisp intersecting hypergraph. Therefore, $\chi(\widetilde{H})^{\langle r_m,s_m\rangle} \leq 3$ since "If H is a crisp intersecting hypergraph, then $\chi(H) \leq 3$.",

Since H is ordered, $\widetilde{H}$ is also ordered. A coloring of $(\widetilde{H})^{\langle r_m,s_m\rangle}$ must be a primitive coloring of $\widetilde{H}$, it follows from known theorem "If H is an ordered IFDHG and A is a primitive coloring of H , then A is a $\mathcal{K}$ -coloring of H" that a coloring of $(\widetilde{H})^{\langle r_m,s_m\rangle}$ is a $\mathcal{K}$ -coloring of $\widetilde{H}$. Therefore, $\chi(\widetilde{H}) \leq 3$ implies that $\chi(H) \leq 3$.

Corollary 8.3.5. If IFDHG H is elementary and essentially intersecting, then $\chi(H) \leq 3$.

Corollary 8.3.6. If H is (μ, ν)-tempered IFDHG and essentially intersecting, then $\chi(H) \leq 3$.

Definition 8.3.7. An IFDHG is said to be *non-trivial* if it has at least one edge E such that $|supp(E)| \geq 2$.

Definition 8.3.8. An IFDHG H is said to be *sequentially simple* if $C(H) = \{H^{\langle r_i,s_i\rangle} = (X^{\langle r_i,s_i\rangle}, \mathcal{E}^{\langle r_i,s_i\rangle}) | \langle r_i, s_i\rangle \in F(H)\}$ satisfies the property that if $E \in \mathcal{E}^{\langle r_{i+1},s_{i+1}\rangle} \setminus \mathcal{E}^{\langle r_i,s_i\rangle}$, then $E \nsubseteq X^{\langle r_i,s_i\rangle}$, where $r_n < ... < r_1, s_n < ... < s_1$. H is said to be *essentially sequentially simple* if $\widetilde{H}$ is sequentially simple.

Definition 8.3.9. Suppose $H = \{E_i \in \mathcal{IF}_p(V) | i = 1, 2, 3, ..., m\}$ is a finite collection of intuitionistic fuzzy subsets of V and let $r, s \in (0, 1]$. Then $H|_{\langle r,s \rangle} = \{E \in \mathcal{IF}_p(V) | h(E) = \langle r, s \rangle\}$ is the set of edges of height $\langle r, s \rangle$. In particular, $H|_{\langle r,s \rangle}$ is the partial directed hypergraph of $H = (V, \mathcal{E})$ with edgeset $\mathcal{E}|_{\langle r,s \rangle}$, provided $\mathcal{E}|_{\langle r,s \rangle} \neq \phi$.

Definition 8.3.10. Let $H_i = (X_i, \mathcal{E}_i), i = 1, 2$ be IFDHGs. Then $H_1 \preceq H_2$ if every edge of H_1 contains an edge of H_2.

Theorem 8.3.11. An IFDHG H is strongly intersecting if and only if $H^{\langle r_i, s_i \rangle} \preceq Tr(H^{\langle r_i, s_i \rangle})$ for every $H^{\langle r_i, s_i \rangle} \in C(H)$.

Proof: By Theorem 8.2.1, known definition and lemma "A crisp hypergraph H is intersecting if and only if $H \preceq Tr(H)$", H is strongly intersecting. $\Leftrightarrow$ H is $\mathcal{K}$-intersecting. $\Leftrightarrow$ $H^{\langle r_i, s_i \rangle}$ is intersecting for all $H^{\langle r_i, s_i \rangle} \in C(H)$ $\Leftrightarrow$ $H^{\langle r_i, s_i \rangle} \preceq Tr(H^{\langle r_i, s_i \rangle})$ for all $H^{\langle r_i, s_i \rangle} \in C(H)$.

Theorem 8.3.12. H is a strongly intersecting IFDHG if and only if for every $\langle r_i, s_i \rangle \in F(H)$, $(H^{\langle r_i, s_i \rangle})|_{\langle r_i, s_i \rangle} \preceq Tr(H^{\langle r_i, s_i \rangle})$.

Proof:

Suppose for every $\langle r_i, s_i \rangle \in F(H)$, $(H^{\langle r_i, s_i \rangle})|_{\langle r_i, s_i \rangle} \preceq Tr(H^{\langle r_i, s_i \rangle})$. For each $H^{\langle r_i, s_i \rangle} \in C(H)$, the edge set $E(H^{\langle r_i, s_i \rangle}) = \{\gamma^{\langle r_i, s_i \rangle} | \gamma \in (H^{\langle r_i, s_i \rangle})|_{\langle r_i, s_i \rangle} \preceq \{\tau^{\langle r_i, s_i \rangle} | \tau \in Tr(H^{\langle r_i, s_i \rangle})\} = Tr(E(H^{\langle r_i, s_i \rangle}))$. Hence, $H^{\langle r_i, s_i \rangle} \preceq Tr(H^{\langle r_i, s_i \rangle})$, for every $H^{\langle r_i, s_i \rangle} \in C(H)$ and by Theorem 8.2.1, H is strongly intersecting.

Conversely, suppose H is strongly intersecting. Let $\gamma \in H|_{\langle r_1, s_1 \rangle}$, where $\langle r_1, s_1 \rangle$ is the largest member of $F(H)$. Let $H^{\langle r_j, s_j \rangle} \in C(H)$. To show that $\gamma^{\langle r_j, s_j \rangle}$ is a transversal of $H^{\langle r_j,s_j\rangle}$ For suppose $E \in H^{\langle r_j,s_j\rangle}$. Then there is an edge η of H such that $\eta^{\langle r_j,s_j\rangle} = E$. Since H is strongly intersecting, there is a spike σ_x such that $h(\sigma_x) = h(\gamma) \wedge h(\eta) = h(\eta) \geq \langle r_j, s_j\rangle$, and support $\{x\}$, which is contained in both γ and η.

Hence, $x \in E \cap \alpha^{\langle r_j,s_j\rangle}$. Thus, γ is a transversal of H and therefore contains a member of $Tr(H)$. Therefore, $(H^{\langle r_i,s_i\rangle})|_{\langle r_i,s_i\rangle} \preceq Tr(H^{\langle r_1,s_1\rangle})$. Using Theorem 8.2.1, H is $\mathcal{K}$-intersecting. Consequently, by Theorem 8.2.1, it follows that $H^{\langle r_i,s_i\rangle}$ must be strongly intersecting. Hence $(H^{r_i,s_i})|_{\langle r_i,s_i\rangle} \preceq Tr(H^{\langle r_i,s_i\rangle})$, for each $\langle r_i, s_i\rangle \in F(H)$.

Corollary 8.3.13. Let H be an IFDHG with $C(H) = \{H^{\langle r_i,s_i\rangle} | \langle r_i, s_i\rangle \in F(H)\}$. Then $H^{\langle r_i,s_i\rangle} \preceq Tr(H^{\langle r_i,s_i\rangle})$, for every $H^{r_i,s_i} \in C(H)$ if and only if $(H^{r_i,s_i})|_{\langle r_i,s_i\rangle} \preceq Tr(H^{\langle r_i,s_i\rangle})$, for every $\langle r_i, s_i\rangle \in F(H)$.

Theorem 8.3.14. An IFDHG H is strongly intersecting if and only if $H^{\langle r_i,s_i\rangle}$ is intersecting for every $\langle r_i, s_i\rangle \in F(H)$.

Proof: By applying the Theorem,

"Let H be an IFDHG and suppose $C(H) = \left\{H^{\langle r_1,s_1\rangle}, H^{\langle r_2,s_2\rangle}, ..., H^{\langle r_n,s_n\rangle}\right\}$. Then H is intersecting if and only if $H^{\langle r_n,s_n\rangle} = (V^{\langle r_n,s_n\rangle}, \mathcal{E}^{\langle r_n,s_n\rangle})$ is intersecting" to $H^{\langle r_i,s_i\rangle}$, and by Theorem 8.2.1, $H^{\langle r_i,s_i\rangle}$ is intersecting for every $\langle r_i, s_i\rangle \in F(H)$ $\Leftrightarrow E(H^{\langle r_i,s_i\rangle})$ is intersecting for each $H^{\langle r_i,s_i\rangle} \in C(H) \Leftrightarrow H$ is $\mathcal{K}$-intersecting $\Leftrightarrow H$ is strongly intersecting.

8.3.2 Application of IFDHG in Chemistry

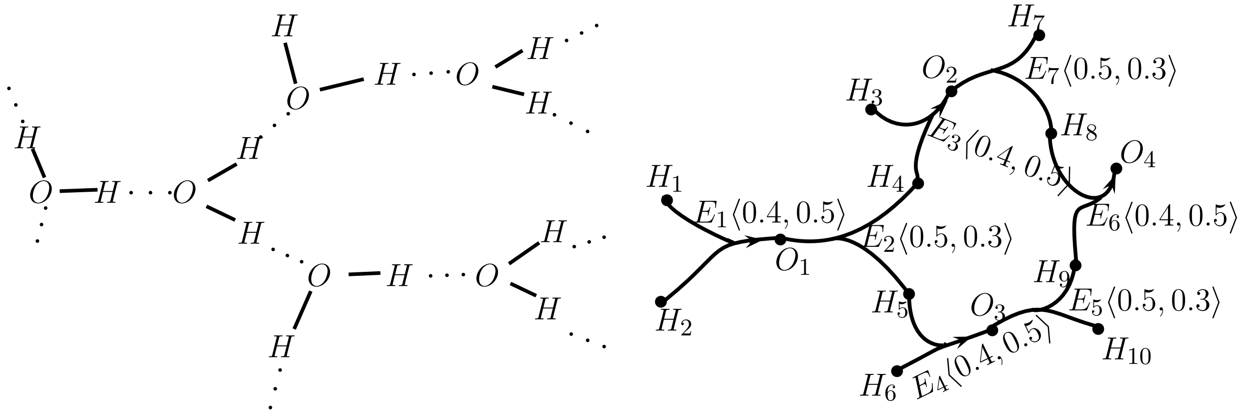
Chemical compounds are formed by the joining of two or more atoms. A *chemical bond* is a lasting attraction between atoms that enables the formation of chemical compounds [29]. There are two major chemical bond classifications namely Primary (Strong) bonds and Secondary (Weak) bonds each with identifiable subgroups as ionic, covalent, metallic and Hydrogen, Van der Waal's bonds respectively.

The power of an atom in a molecule to attract electrons to itself is called *electronegativity*. Covalent bonds are formed when the electronegativity difference (D^c) between the atoms is < 1.7. Ionic bonds are formed when the electronegativity difference (D^c) between the atoms is > 1.7. Based on Pauling scale for Electronegativity, Carbon (C) atom has electonegativity 2.5, Oxygen (O) has 3.5 and Hydrogen (H) has electronegativity 2.1.

Bond length is the distance between centers of atoms bonded within a molecule. Bond length depends on three main factors such as size of atoms, bond strength and multiplicity of bonds. Also, the temperature and pressure affect the bondlength between atoms and hence, uncertainty exists in the molecular structure. therefore, the concept of IFDHG can also be used as a tool to deal this kind of uncertainity.

An IFDHG $H = (V, \mathcal{E})$ is used to represent molecular structure, where $x \in V$ corresponds to an atom, intuitionistic fuzzy directed hyperedges correspond to bonds between the atoms. Such IFDHGs are known as molecular IFDHGs. The directions of intuitionistic fuzzy hyperedges represent the direction towards the atom which has more electronegativity. Membership amd non-membership values of the intuitionistic fuzzy hyperedges depends on the length of the bonds between the atom. Bond length depends on bond order between atoms, electronegativity force of the atoms and intermolecular forces between the molecules.

In Figure 8.2(a), the molecular structure of water is shown. Here, the dotted



(a) Molecular structure of water (b) Molecular IFDHG of water

Figure 8.2

lines represent the hydrogen bonds between the Oxygen and Hydrogen atoms, remaining are covalent bonds. In Figure 8.2(b), molecular IFDHG representation of water is shown. In this molecular IFDHG, the directions represent the direction towards the atom which has more electronegativity. Intuitionistic fuzzy directed hyperedge E_1 connect two Hydrogen atoms with an Oxygen atom. Oxygen atom has more eletronegativity than the Hydrogen atom. So the $hd(E_1)$ is Oxygen atom and two hydrogen atoms are $tl(E_1)$.

The membership and non-membership values of $E_i, i = 1, ..., 7$ is denoted by $\langle \mu(E_i), \nu(E_i)\rangle$. The bond length of the covalent bond between Hydrogen and Oxygen atoms is $0.96A^0$(Angstrom) and hydrogen bond length between these two atoms is $1.97A^0$(Angstrom).

In intuitionistic fuzzy triangular function, let $a = 0.5, b = 1.5$ and $c = 3.0, x = 0.96$ (Bond length). Therefore, $\langle\mu(E_i), \nu(E_i)\rangle = \langle 0.4, 0.5\rangle$ for $i = 1, 3, 4$ and 6. In a similar way, the membership and non-membership values of intuitionistic fuzzy hyperedges are calculated.

Chapter 9

Multi-parameter temporal intuitionistic fuzzy sets

9.1 Introduction

Fuzzy sets (FSs) introduced by L.A.Zadeh in 1965 [110] are generalization of crisp sets. FSs which incorporate the partial membership of the elements of the set showed meaningful applications in several fields like science, engineering, medicine etc. K.T.Atanassov introduced the concept of intuitionistic fuzzy sets (IFSs) in 1983 [7] as an extension of FSs. These sets include not only the membership of the element in the set but also the non-membership of the element along with degree of hesitancy. K.T.Atanassov also extended the concept of IFSs into temporal intuitionistic fuzzy sets (TIFSs) [9]. TIFSs give a possibility to trace the changes of the object considered for all the time moments from a time scale and permit more detailed estimations of the real time processes flowing in time. Intuitionistic fuzzy multi-dimensional sets (IFMDSs) were introduced and described in [4-8] as extensions of TIFSs. Membership functions convert crisp into fuzzy values within the system. Depending upon the model, special type of membership functions which take the shape of triangles, trapezoids, bell curves etc. can be chosen for consideration. In case of IFSs, both membership and non-membership functions are required to convert crisp into intuitionistic fuzzy values within the system. For the sake of convenience, the term *'intuitionistic fuzzification functions'* is used to denote membership and non-membership functions through out this chapter. The rest of the chapter is organized as follows. In Section 2, a short review of the basic definitions regarding FSs, IFSs, TIFSs and IFMDSc are given. Multi-parameter temporal intuitionistic fuzzy sets (MTIFSs) are proposed as generalization of IFMDSs and TIFSs and a few relations and operations are defined on them in Section 3. Extended triangular intuitionistic fuzzification functions of a TIFS and MTIFS are defined in Section 4 and geometric interpretation of a TIFS is shown in Section 5 with an illustration.

9.2 Preliminaries

In this section, a concise overview of the basic definitions related to FSs, IFSs and TIFSs are presented.

Definition 9.2.1. [9] Let E be the universe and T be a non-empty set of time moments. Then, a *temporal intuitionistic fuzzy set* (TIFS) is an object having the form

$$A(T) = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle / (x,t) \in E \times T\}$$

where

(i) $A \subset E$ is a fixed set.

(ii) $\mu_A(x,t)$ and $\nu_A(x,t)$ denote the degrees of membership and non-membership respectively of the element (x,t) such that $0 \leq \mu_A(x,t) + \nu_A(x,t) \leq 1$ for all $(x,t) \in E \times T$

Definition 9.2.2. [17] Let the sets $Z_1, Z_2, \ldots, Z_n$ be fixed and let for each i $(1 \leq i \leq n) : z_i \in Z_i$. Let the set E be fixed. An IFMDS A in $E \times Z_1 \times Z_2 \times \cdots \times Z_n$ is an object of the form

$A(Z_1, Z_2, \ldots, Z_n) = \{\langle x, \mu_A(x, z_1, z_2, \ldots, z_n), \nu_A(x, z_1, z_2, \ldots, z_n)\rangle / (x, z_1, z_2, \ldots, z_n) \in E \times Z_1 \times Z_2 \times \cdots \times Z_n\}$

where

(i) $\mu_A(x, z_1, z_2, \ldots, z_n) + \nu_A(x, z_1, z_2, \ldots, z_n) \leq 1$ for every $(x, z_1, z_2, \ldots, z_n) \in E \times Z_1 \times Z_2 \times \cdots \times Z_n$.

(ii) $\mu_A(x, z_1, z_2, \ldots, z_n)$ and $\nu_A(x, z_1, z_2, \ldots, z_n)$ are the degrees of membership and non-membership respectively, of the element $(x, z_1, z_2, \ldots, z_n) \in E \times Z_1 \times Z_2 \times \cdots \times Z_n$.

9.3 Multi-parameter Temporal Intuitionistic Fuzzy Sets

The problems occurring in real life are not only uncertain but they often involve distinct set of parameters. These are dealt with the IFMDSs described in [21]-[25]. To meet out the situation when the system is also dynamic, in addition to time in the TIFSs, a set of parameters are are introduced in multi-parameter temporal intuitionistic fuzzy sets (MTIFSs). This set is derived as a special case of IFMDSs introduced in [17].

Definition 9.3.1. Let E be the Universe, T be a non-empty set of time moments and $P = (P_1, P_2, \cdots P_n)$,

$P_i,\ i = 1, 2, \ldots, n$ are distinct sets of parameters on which E depends. Let p be an $n-$tuple $(p_1, p_2, \cdots p_n)$, where $p_i \in P_i,\ i = 1, 2, \ldots, n$. Then *a multi- parameter temporal intuitionistic fuzzy set* (MTIFS) defined on E is an object of the form

$$A(T, P) = \left\{\langle x, \mu_A(x, t, p), \nu_A(x, t, p)\rangle / (x, t, p) \in E \times T \times \prod_{i=1}^{n} P_i\right\}$$

where

(i) $A \subset E$ is a fixed set.

(ii) $\mu_A(x, t, p)$ and $\nu_A(x, t, p)$ denote the degrees of membership and non-membership respectively of the element $(x, t, p) \in E \times T \times \prod_{i=1}^{n} P_i$ such that $0 \leq \mu_A(x, t, p) + \nu_A(x, t, p) \leq 1$ for all $(x, t, p) \in E \times T \times \prod_{i=1}^{n} P_i$.

Note:

(i) Multi-parameter TIFS is a TIFS when $P = (\phi, \phi \ldots, \phi)$.

(ii) Multi-parameter TIFS is an IFS when $P = (\phi, \phi \ldots, \phi)$ and T is a singleton set.

(iii) Multi-parameter TIFS is an IFMDS when T is a singleton set.

(iv) The notations $P = (P_1, P_2, \cdots P_n)$ and $p = (p_1, p_2, \cdots p_n)$, where $p_i \in P_i$ are used through out this paper.

(v) Each MTIFS is a IFMDS for which $Z_1 = T, Z_2 = P_1, Z_3 = P_2, \cdots Z_{n+1} = P_n$. The present form of MTIFS is more suitable for the investigation of dynamical processes in which the time-component is very important and plays a central role.

Definition 9.3.2. The operators C^* and I^* over a MTIFS are defined as follows:

$$
\begin{aligned}
C^*\left(A\left(T,P\right)\right) &= \left\{\left\langle x, \max_{t\in T}\mu_A(x,t,p), \min_{t\in T}\nu_A(x,t,p)\right\rangle / x\in E\right\}\\
I^*\left(A\left(T,P\right)\right) &= \left\{\left\langle x, \min_{t\in T}\mu_A(x,t,p), \max_{t\in T}\nu_A(x,t,p)\right\rangle / x\in E\right\}
\end{aligned}
$$

The following results are derived from IFMDSs [21]-[25].

Theorem 9.3.3. For every MTIFS $A(T,P), C^*(A(T,P))$ and $I^*(A(T,P))$ are MTIFSs.

Proof. Let $\max\limits_{t\in T}\mu_A(x,t,p) = \mu_A(x,t',p)$ for some $t'\in T$ and
$\min\limits_{t\in T}\nu_A(x,t,p) = \nu_A(x,t'',p)$ for some $t''\in T$.
Then, $\nu_A(x,t'',p)\leq \nu_A(x,t',p)$ and

$$
\begin{aligned}
\max_{t\in T}\mu_A(x,t,p)+\min_{t\in T}\nu_A(x,t,p) &= \mu_A(x,t',p)+\nu_A(x,t'',p)\\
&\leq \mu_A(x,t',p)+\nu_A(x,t',p)\\
&\leq 1
\end{aligned}
$$

Hence $C^*(A(T,P))$ is a MTIFS. Also, $I^*(A(T,P))$ is a MTIFS can be proved in a similar manner. $\square$

Theorem 9.3.4. For every MTIFS $A(T,P)$,

$$
C^*\left(C^*(A(T,P))\right) = C^*(A(T,P))
$$

$$
\begin{aligned}
C^*\left(I^*(A(T,P))\right) &= I^*(A(T,P)) \\
I^*\left(C^*(A(T,P))\right) &= C^*(A(T,P)) \\
I^*\left(I^*(A(T,P))\right) &= I^*(A(T,P))
\end{aligned}
$$

Theorem 9.3.5. For every MTIFS $A(T,P)$,

$$
\begin{aligned}
C\left(C^*(A(T,P))\right) &= C^*(C(A(T,P))) \\
I\left(I^*(A(T,P))\right) &= I^*(I(A(T,P)))
\end{aligned}
$$

Proof.

$$
\begin{aligned}
C(C^*(A(T,P)) &= C\left(\left\{\left\langle x, \max_{t\in T}\mu_A(x,t,p), \min_{t\in T}\nu_A(x,t,p)\right\rangle / x\in E\right\}\right) \\
&= \left\{\left\langle x, \max_{x\in E}\max_{t\in T}\mu_A(x,t,p), \min_{x\in E}\min_{t\in T}\nu_A(x,t,p)\right\rangle / x\in E\right\} \\
&= \left\{\left\langle x, \max_{t\in T}\max_{x\in E}\mu_A(x,t,p), \min_{t\in T}\min_{x\in E}\nu_A(x,t,p)\right\rangle / x\in E\right\} \\
&= C^*(C(A(T,P))) \\
I(I^*(A(T,P)) &= I\left(\left\{\left\langle x, \min_{t\in T}\mu_A(x,t,p), \max_{t\in T}\nu_A(x,t,p)\right\rangle / x\in E\right\}\right) \\
&= \left\{\left\langle x, \min_{x\in E}\min_{t\in T}\mu_A(x,t,p), \max_{x\in E}\max_{t\in T}\nu_A(x,t,p)\right\rangle / x\in E\right\} \\
&= \left\{\left\langle x, \min_{t\in T}\min_{x\in E}\mu_A(x,t,p), \max_{t\in T}\max_{x\in E}\nu_A(x,t,p)\right\rangle / x\in E\right\} \\
&= I^*(I(A(T,P)))
\end{aligned}
$$

$\square$

Theorem 9.3.6. For every two MTIFSs $A(T', P)$ and $B(T'', P)$,

$$\begin{aligned}
C^*(A(T', P) \cap B(T'', P)) &\subset C^*(A(T', P)) \cap C^*(B(T'', P)) \\
C^*(A(T', P) \cup B(T'', P)) &= C^*(A(T', P)) \cup C^*(B(T'', P)) \\
I^*(A(T', P) \cap B(T'', P)) &= I^*(A(T', P)) \cap C^*(B(T'', P)) \\
I^*(A(T', P) \cup B(T'', P)) &\supset I^*(A(T', P)) \cup I^*(B(T'', P))
\end{aligned}$$

9.3.1 Basic relations and operations on MTIFSs

Let E be the Universe. $T^{'}$ and $T^{''}$ are any two non-empty sets of time moments and $P = (P_1, P_2, \cdots P_n)$ are distinct sets of parameters on which E depends . Let $A(T^{'}, P)$ and $B(T^{''}, P)$ are any two MTIFSs defined as follows.

$$A(T^{'}, P) = \left\{ \langle x, \mu_A(x,t,p), \nu_A(x,t,p) \rangle / (x,t,p) \in E \times T^{'} \times \prod_{i=1}^{n} P_i \right\}$$

and

$$B(T^{''}, P) = \left\{ \langle x, \mu_B(x,t,p), \nu_B(x,t,p) \rangle / (x,t,p) \in E \times T^{''} \times \prod_{i=1}^{n} P_i \right\}$$

Let

$$\overline{\mu_A}(x,t,p) = \begin{cases} \mu_A(x,t,p), & t \in T^{'} \\ 0, & t \in T^{''} - T^{'} \end{cases}$$

$$\overline{\mu_B}(x,t,p) = \begin{cases} \mu_B(x,t,p), & t \in T^{''} \\ 0, & t \in T^{'} - T^{''} \end{cases}$$

$$\overline{\nu_A}(x,t,p) = \begin{cases} \nu_A(x,t,p), & t \in T^{'} \\ 1, & t \in T^{''} - T^{'} \end{cases}$$

$$\overline{\nu_B}(x,t,p) = \begin{cases} \nu_B(x,t,p), & t \in T^{''} \\ 1, & t \in T^{'} - T^{''} \end{cases}$$

Then, the basic set operations on the two sets $A(T^{'},P)$ and $B(T^{''},P)$ are defined as follows.

1. *Inclusion* $(T^{'} = T^{''} = T)$ $A(T,P) \subset B(T,P)$ iff $\mu_A(x,t,p) \leq \mu_B(x,t,p)$ and $\nu_A(x,t,p) \geq \mu_B(x,t,p) \forall \quad (x,t,p) \in E \times T \times \prod_{i=1}^{n} P_i$.

2. *Equality* $(T^{'} = T^{''} = T)$

   $A(T,P) = B(T,P)$ iff $\mu_A(x,t,p) = \mu_B(x,t,p)$ and

   $\nu_A(x,t,p) = \nu_B(x,t,p) \quad \forall \quad (x,t,p) \in E \times T \times \prod_{i=1}^{n} P_i$.

3. *Complement* $\bar{A}(T,P) = \left\{ \langle x, \nu_A(x,t,p), \mu_A(x,t,p) \rangle / (x,t,p) \in E \times T \times \prod_{i=1}^{n} P_i \right\}$.

4. *Intersection*

   $A(T^{'},P) \cap B(T^{''},P) = \{\langle x, \min(\overline{\mu_A}(x,t,p), \overline{\mu_B}(x,t,p)), \max(\overline{\nu_A}(x,t,p), \overline{\nu_B}(x,t,p)) \rangle\}$

   where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

5. *Union* $A(T^{'},P) \cup B(T^{''},P) = \{\langle x, max(\overline{\mu_A}(x,t,p), \overline{\mu_B}(x,t,p)), min(\overline{\nu_A}(x,t,p), \overline{\nu_B}(x,t,p)) \rangle\}$

   where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

6. *Addition* $A(T',P) \oplus B(T'',P) =$

   $\{\langle x, (\overline{\mu_A}(x,t,p) + \overline{\mu_B}(x,t,p) - \overline{\mu_A}(x,t,p).\overline{\mu_B}(x,t,p), (\overline{\nu_A}(x,t,p).\overline{\nu_B}(x,t,p)) \rangle\}$

   where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

7. *Multiplication* $A(T', P) \otimes B(T'', P) =$

   $\{\langle x, (\overline{\mu_A}(x,t,p).\overline{\mu_B}(x,t,p)), \overline{\nu_A}(x,t,p) + \overline{\nu_B}(x,t,p) - (\overline{\nu_A}(x,t,p).\overline{\nu_B}(x,t,p)\rangle\}$

   where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

8. *Averaging Operator* $A(T', P)@B(T'', P) = \Big\{\langle x, \frac{1}{2}(\overline{\mu_A}(x,t,p) + \overline{\mu_B}(x,t,p)), \frac{1}{2}(\overline{\nu_A}(x,t,p) + \overline{\nu_B}(x$

   where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

9. $A(T', P) \oslash B(T'', P) = \Big\{\langle x, \sqrt{\overline{\mu_A}(x,t,p).\overline{\mu_B}(x,t,p)}, \sqrt{\overline{\nu_A}(x,t,p).\overline{\nu_B}(x,t,p)}\rangle\Big\}$

   where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

10. $A(T', P) * B(T'', P) = \Big\{\langle x, \frac{\overline{\mu_A}(x,t,p)+\overline{\mu_B}(x,t,p)}{2(\overline{\mu_A}(x,t,p).\overline{\mu_B}(x,t,p)+1)}, \frac{\overline{\nu_A}(x,t,p)+\overline{\nu_B}(x,t,p)}{2(\overline{\nu_A}(x,t,p).\overline{\nu_B}(x,t,p)+1)}\rangle\Big\}$

    where $(x,t,p) \in E \times (T' \cup T'') \times \prod_{i=1}^{n} P_i$.

9.3.2 Algebraic Laws in MTIFSs

Let A, B and C are any three MTIFSs defined on E , then the following algebraic laws hold good.

1. $(A^c)^c = A$ (complementary law)

2. (i) $A \cup A = A$

   (ii) $A \cap A = A$(idempotent laws).

3. (i) $A \cup B = B \cup A$

   (ii) $A \cap B = B \cap A$ (commutative laws)

4. (i) $(A \cup B) \cup C = A \cup (B \cup C)$

   (ii) $(A \cap B) \cap C = A \cap (B \cap C)$(associative laws)

5. (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributive laws). Right distributive laws also holds.

6. (i) $(A \cup B)^c = A^c \cap B^c$

   (ii) $(A \cap B)^c = A^c \cup B^c$(De morgan's laws)

7. (i) $A \cap (A \cup B) = A$

   (ii) $A \cup (A \cap B) = A$ (absorption laws).

8. (i) $A \oplus B = B \oplus A$

   (ii) $A \otimes B = B \otimes A$

9. (i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

   (ii) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

10. (i) $(A \oplus B)^c = (A)^c \otimes (B)^c$

    (ii) $(A \otimes B)^c = (A)^c \oplus (B)^c$

11. (i) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$

    (ii) $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$

    (iii) $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$

    (iv) $A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C)$

9.4 Triangular intuitionistic fuzzification functions for TIFS and MTIFS

Fuzzification plays an important role in fuzzy logic controllers. Designing intuitionistic fuzzification functions is necessary to develop intuitionistic fuzzy logic controller system for a TIFS and MTIFS. In this section, extended triangular intuitionistic fuzzification functions for TIFS and MTIFS are established.

Definition 9.4.1. Let E be the Universe, T be a non-empty set of time moments and $A(T)$ be the corresponding temporal intuitionistic fuzzy set defined on E. Then, the *extended triangular intuitionistic fuzzification functions* for the TIFS $A(T)$ are defined as

$$\mu_A(x,t) = \begin{cases} \frac{x+t-(a+c)}{m+m'-(a+c)} & a \leq x \leq m, c \leq t \leq m' \\ \frac{x+t-(a+m')}{m+d-(a+m')} & a \leq x \leq m, m' < t \leq d \\ \frac{b+m'-(x+t)}{b+m'-(m+c)} & m < x \leq b, c \leq t \leq m' \\ \frac{b+d-(x+t)}{b+d-(m+m')} & m < x \leq b, m' < t \leq d \\ 0 & otherwise \end{cases},$$

$$\nu_A(x,t) = \begin{cases} \frac{m+m'-(x+t)}{m+m'-(a+c)} & a \leq x \leq m, c \leq t \leq m' \\ \frac{m+d-(x+t)}{m+d-(a+m')} & a \leq x \leq m, m' < t \leq d \\ \frac{x+t-(m+c)}{b+m'-(m+c)} & m < x \leq b, c \leq t \leq m' \\ \frac{x+t-(m+m')}{b+d-(m+m')} & m < x \leq b, m' < t \leq d \\ 1 & otherwise \end{cases}$$

where $a \leq x \leq b$; $a < m < b$; $c \leq t \leq d$ and $c < m' < d$.

Definition 9.4.2. Let E be the universe, T be a non empty set of time moments and P be a set of parameters on which E depends and $A(T, P)$ be the MTIFS defined on E. Then *the extended triangular intuitionistic fuzzification functions* for the MTIFS $A(T, P)$ are defined as

$$\mu_A(x,t,p) = \begin{cases} \frac{x+t+\sum p_i-(a+c+\sum e_i)}{m+m'+\sum m_i''-(a+c+\sum e_i)} & a \leq x \leq m, c \leq t \leq m', e_i \leq p_i \leq m_i'' \\ . \\ . \\ . \\ \frac{b+d+\sum f_i-(x+t+\sum p_i)}{b+d+\sum f_i-(m+m'+\sum m_i'')} & m < x \leq b, m' < t \leq d, m_i'' < p_i \leq f_i \\ 0 \quad otherwise \end{cases}$$

and

$$\nu_A(x,t,p) = \begin{cases} \frac{m+m'+\sum m_i''-(x+t+\sum p_i)}{m+m'+\sum m_i''-(a+c+\sum e_i)} & a \leq x \leq m, c \leq t \leq m', e_i \leq p_i \leq m_i'' \\ . & \\ . & \\ . & \\ \frac{x+t+\sum p_i-(m+m'+\sum m_i'')}{b+d+\sum f_i-(m+m'+\sum m_i'')} & m < x \leq b, m' < t \leq d, m_i'' < p_i \leq f_i \\ 1 \quad otherwise \end{cases}$$

where

$a \leq x \leq b$; $a < m < b$; $c \leq t \leq d$; $c < m' < d$; $e_i \leq p_i \leq f_i$ and $e_i < m_i'' < f_i$, $i = 1, 2, \ldots, n$.

Special Case $(P = P_1)$

Extended triangular intuitionistic fuzzification functions for a single parameter set are as follows:

$$\mu_A(x,t,p) = \begin{cases} \frac{x+t+p-(a+c+e)}{m+m'+m''-(a+c+e)} & a \le x \le m, c \le t \le m', e \le p \le m'' \\ \frac{x+t+p-(a+c+m'')}{m+m'+f-(a+c+m'')} & a \le x \le m, c \le t \le m', m'' < p \le f \\ \frac{x+t+p-(a+m'+e)}{m+d+m''-(a+m'+e)} & a \le x \le m, m' < t \le d, e \le p \le m'' \\ \frac{x+t+p-(a+m'+m'')}{m+d+f-(a+m'+m'')} & a \le x \le m, m' < t \le d, m'' < p \le f \\ \frac{b+m'+m''-(x+t+p)}{b+m'+m''-(m+c+e)} & m < x \le b, c \le t \le m', e \le p \le m'' \\ \frac{b+m'+f-(x+t+p)}{b+m'+f-(m+c+m'')} & m < x \le b, c \le t \le m', m'' < p \le f \\ \frac{b+d+m''-(x+t+p)}{b+d+m''-(m+m'+e)} & m < x \le b, m' < t \le d, e \le p \le m'' \\ \frac{b+d+f-(x+t+p)}{b+d+f-(m+m'+m'')} & m < x \le b, m' < t \le d, m'' < p \le f \\ 0 \quad otherwise \end{cases}$$

$$\nu_A(x,t,p)=\begin{cases}\frac{m+m'+m''-(x+t+p)}{m+m'+m''-(a+c+e)} & a\leq x\leq m, c\leq t\leq m', e\leq p\leq m''\\ \frac{m+m'+f-(x+t+p)}{m+m'+f-(a+c+m'')} & a\leq x\leq m, c\leq t\leq m', m''<p\leq f\\ \frac{m+d+m''-(x+t+p)}{m+d+m''-(a+m'+e)} & a\leq x\leq m, m'<t\leq d, e\leq p\leq m''\\ \frac{m+d+f-(x+t+p)}{m+d+f-(a+m'+m'')} & a\leq x\leq m, m'<t\leq d, m''<p\leq f\\ \frac{x+t+p-(m+c+e)}{b+m'+m''-(m+c+e)} & m<x\leq b, c\leq t\leq m', e\leq p\leq m''\\ \frac{x+t+p-(m+c+m'')}{b+m'+f-(m+c+m'')} & m<x\leq b, c\leq t\leq m', m''<p\leq f\\ \frac{x+t+p-(m+m'+e)}{b+d+m''-(m+m''+e)} & m<x\leq b, m'<t\leq d, e\leq p\leq m''\\ \frac{x+t+p-(m+m'+m'')}{b+d+f-(m+m'+m'')} & m<x\leq b, m'<t\leq d, m''<p\leq f\\ 1 \quad otherwise & \end{cases}$$

where $a \leq x \leq b$; $a < m < b$; $c \leq t \leq d$; $c < m' < d$; $e \leq p \leq f$ and $e < m'' < f$.

9.5 Geometric representation of the extended triangular intuitionistic fuzzification functions of a TIFS

In this section, geometric representation of the extended triangular intuitionistic fuzzification functions of a TIFS given in Definition 9.4.1 are discussed with an illustration. Consider a TIFS X={10,20,30,40,50} with the time domain T={1,2,3}. Then ,the extended triangular intuitionistic fuzzification values are calculated and their pictorial representation are shown in Figures 9.1-9.4.

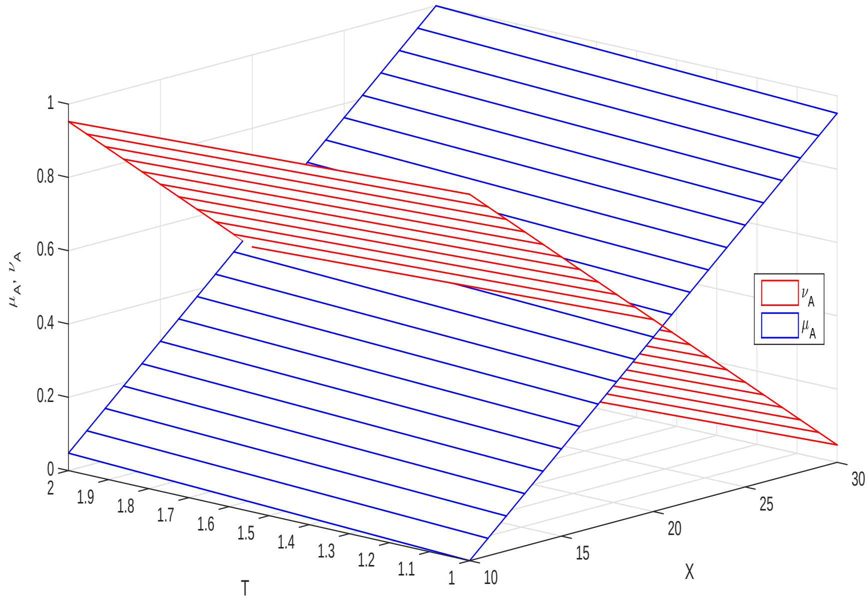


Figure 9.1: Membership and non-membership functions for $a \leq x \leq m$ & $c \leq t \leq m'$

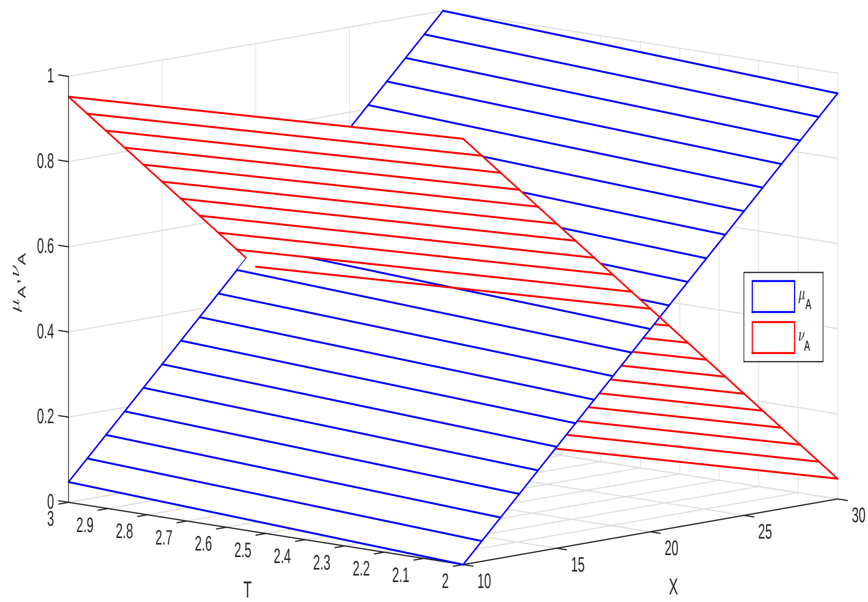


Figure 9.2: Membership and non-membership functions for $a \leq x \leq m$ & $m' < t \leq d$

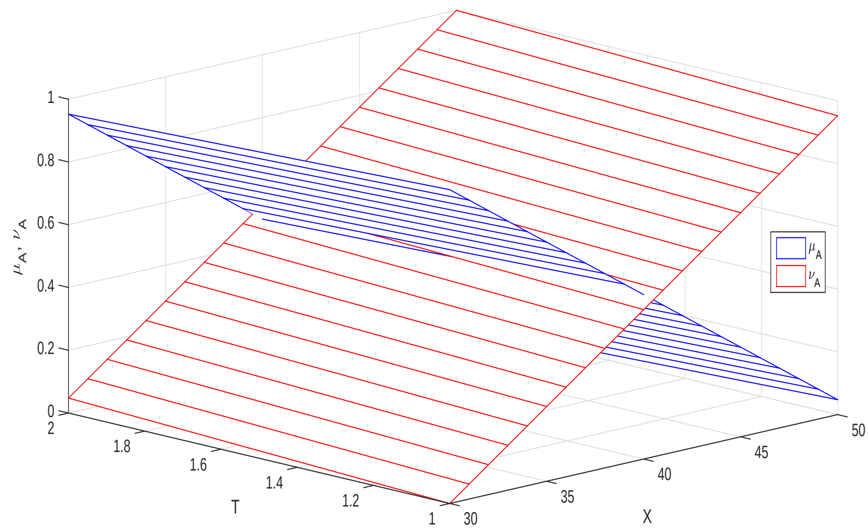


Figure 9.3: Membership and non-membership functions for $m < x \leq b$ & $c \leq t \leq m'$

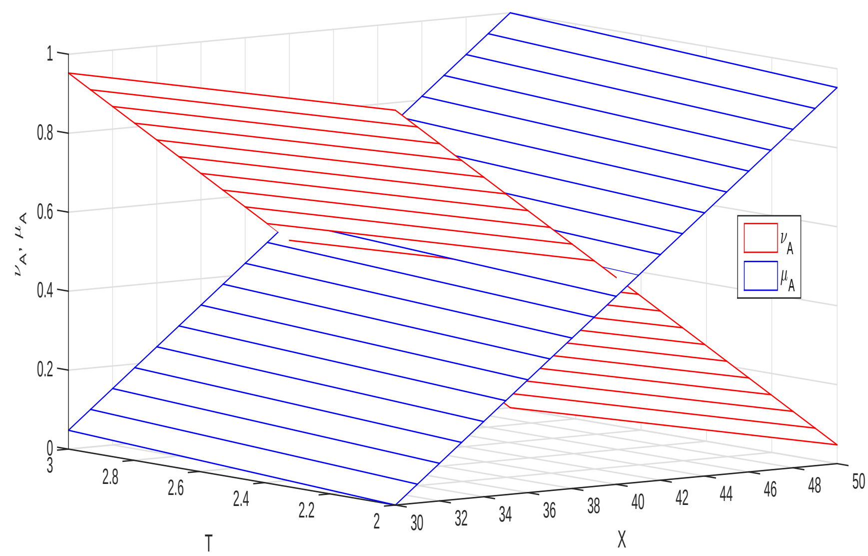


Figure 9.4: Membership and non-membership functions for $m < x \leq b$ & $m' < t \leq d$

Chapter 10

A study on Indian Universities ranking using intercriteria decision making method

10.1 Introduction

In this chapter, the ICDM method is discussed. InterCriteria Analysis, also known as InterCriteria Decision Making, is an approach that takes an index matrix containing evaluations of objects against a set of criteria as input and calculates the degrees of correlation between each pair of the criteria in the form of intuitionistic fuzzy pairs [19]. The InterCriteria Decision Making introduced by K.T.Atanassov, D.Mavrov and V.Atanassova [19] is based on the theory of the intuitionistic fuzzy sets and the index matrices [13]. Using the ICDM method, studies have already been made on university ranking in two countries Bulgaria (2015) and Poland (2016) and published in [31] and [54] respectively. This way, the researcher got motivated to use ICDM in Indian universities' ranking for the year 2017. Hence, an attempt has been made to apply InterCriteria Decision Making (ICDM) method to discuss about the parameters involved in the ratings of Universities in India.

The purpose is to identify the best correlated groups of indicators in the Ranking System for the Indian Universities. By applying the ICDM approach over extracted data, which finds the indicators that have the highest dependencies and to observe their behaviour during the year. This approach helps to determine the precision and confirm the current weights of the indicators.

The real data extracted from Universities Ranking System, that is from the sites of a relevant rating system which provide free access to data.

10.2 Preliminaries

In this section, the preliminary definitions required for the present paper are collected and presented.

Definition 10.2.1. [19] The *intuitionistic fuzzy pair* (IFP) is an object with the form $\langle a, b\rangle$ where $a, b \in [0, 1]$ and $a + b \leq 1$ that is used as an evaluation of some object or process and which components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ be two intuitionistic fuzzy pairs, then the relations are defined as

$$\begin{aligned}
x < y &\quad \text{iff } a < c \text{ and } b > d\\
x \leq y &\quad \text{iff } a \leq c \text{ and } b \geq d\\
x = y &\quad \text{iff } a = c \text{ and } b = d\\
x \geq y &\quad \text{iff } a \geq c \text{ and } b \leq d\\
x > y &\quad \text{iff } a > c \text{ and } b < d.
\end{aligned}$$

Definition 10.2.2. [19] Let I be a fixed set of indices and $\Re$ be the set of all real numbers. Then *index matrix* with index sets K and $L\,(K, L \subset I)$, takes the form

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \ldots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \ldots & a_{k_2,l_n} \\ . & & & & \\ . & & & & \\ . & & & & \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \ldots & a_{k_m,l_n} \end{array}$$

where $K = \{k_1, k_2, ..., k_m\}, L = \{l_1, l_2, ..., l_n\}$ and for $1 \leq i \leq m$ and $1 \leq j \leq n$: $a_{k_i,l_j} \in \Re$.

Definition 10.2.3. [19] The *intuitionistic fuzzy index matrix* (IFIM) - takes the form

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \langle \mu_{k_1,l_2}, \nu_{k_1,l_2} \rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ k_2 & \langle \mu_{k_2,l_1}, \nu_{k_2,l_1} \rangle & \langle \mu_{k_2,l_2}, \nu_{k_2,l_2} \rangle & \ldots & \langle \mu_{k_2,l_n}, \nu_{k_2,l_n} \rangle \\ . & & & & \\ . & & & & \\ . & & & & \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \langle \mu_{k_m,l_2}, \nu_{k_m,l_2} \rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array}$$

where for every $1 \leq i \leq m$, $1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$, i.e., $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$ is an IFP.

10.3 InterCriteria Decision Making Analysis

In this section, method and description of intercriteria decision making (ICDM) method and a study on its uses in the indian universities' ranking system for the year 2017 are presented.

10.3.1 InterCriteria Decision Making (ICDM) method

The ICDM approach helps to discover the relationship and examine the correlation between the indicators used in the Bulgarian university ratings [31]. The idea of InterCriteria Analysis and first steps of this research began in the end of 2013, presented in 12^{th} International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, Warsaw (2013) and published in [19] by K.T.Atanassov. The object can be estimated on the base of several criteria. The number of the criteria can be reduced by calculating the correlations in each pair of criteria in the form of intuitionistic fuzzy pairs of values [21]. The intuitionistic fuzzy pairs of values are the intuitionistic fuzzy evaluation in the interval $[0, 1]$. The relations can be established between any two group of indicators I_k and I_l.

10.3.2 Description of ICDM method

Let U_p be the number of universities, $p = 1, 2, ...m$ and I_q be the number of group of indicators, $q = 1, 2, ..., n$. The aim is to evaluate the universities (objects) with the group of criteria. The index matrix M that contains two sets of indices, one for rows and another for columns [19].

$$M = \begin{array}{c|cccccccc} & I_1 & ... & I_k & ... & I_l & ... & I_n \\ \hline U_1 & a_{U_1,I_1} & ... & a_{U_1,I_k} & ... & a_{U_1,I_l} & ... & a_{U_1,I_n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ U_i & a_{U_i,I_1} & ... & a_{U_i,I_k} & ... & a_{U_i,I_l} & ... & a_{U_i,I_n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ U_j & a_{U_j,I_1} & ... & a_{U_j,I_k} & ... & a_{U_j,I_l} & ... & a_{U_j,I_n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ U_m & a_{U_m,I_1} & ... & a_{U_m,I_k} & ... & a_{U_m,I_l} & ... & a_{U_m,I_n} \end{array}$$

Let U_p is an evaluating object and I_q is an evaluation criteria, for every p, q $(1 \leq$

$p \leq m, 1 \leq q \leq n$). Then a_{U_p,I_q} is the evaluation of the p-th object against the q-th criteria, defined as a real number or another object that is comparable according to relation R with all the rest elements of the index matrix M [19].

Then the InterCriteria Analysis is applied for calculating evaluations. Assume that the relation $R(a_{U_i,I_k}, a_{U_j,I_k})$ holds for each i, j, k for the comparability requirement. Let $\overline{R}$ be the dual relation of R, which is possible only if the relation R is false, and vice versa [19].

The comparision between every two different criteria are made along all the objects are evaluated in pairwise. During the comparision, it is maintained one counter of the number of times when the relation R holds and another counter for the dual relation $\overline{R}$ [19].

The number of cases, in which the relation $R(a_{U_i,I_k}, a_{U_j,I_k})$ and $R(a_{U_i,I_l}, a_{U_j,I_l})$ are simultaneously satisfied, is denoted by $S^{\mu}_{k,l}$ and also the number of cases, in which the relation $R(a_{U_i,I_k}, a_{U_j,I_k})$ and $\overline{R}(a_{U_i,I_l}, a_{U_j,I_l})$ are simultaneously satisfied, is denoted by $S^{\nu}_{k,l}$.

The total number of pairwise comparision between the objects is $\dfrac{m(m-1)}{2}$, then the following inequality holds:

$$0 \leq S^{\mu}_{k,l} + S^{\nu}_{k,l} \leq \frac{m(m-1)}{2}.$$

For every k, l such that $1 \leq k \leq l \leq n$, and for $m \geq 2$, define two numbers as

$$\mu_{I_k,I_l} = 2\frac{S^{\mu}_{k,l}}{m(m-1)}, \quad \nu_{I_k,I_l} = 2\frac{S^{\nu}_{k,l}}{m(m-1)}$$

which result new index matrix M^* with intuitionistic fuzzy pairs $\left\langle \mu_{I_k,I_l}, \nu_{I_k,I_l} \right\rangle$ to represent an intuitionistic fuzzy evaluation of the relations between every pair of criteria I_k and I_l. In this way, the index matrix M that relates evaluating object with evaluation criteria can be transformed to another index matrix M^* that gives the relations among the criteria [19]:

$$M^* = \begin{array}{c|ccc} & I_1 & ... & I_n \\ \hline I_1 & \langle \mu_{I_1,I_1}, \nu_{I_1,I_1} \rangle & ... & \langle \mu_{I_1,I_n}, \nu_{I_1,I_n} \rangle \\ ... & ... & ... & ... \\ I_n & \langle \mu_{I_n,I_1}, \nu_{I_n,I_1} \rangle & ... & \langle \mu_{I_n,I_n}, \nu_{I_n,I_n} \rangle \end{array}$$

The two index matrices M^μ and M^ν are more flexible to work in practical considerations, rather than with the index matrix M^* of IF pairs [54]. The last step of the algorithm is to determine the degrees of correlation between the criteria depending of the choosen threshold for μ and ν.

Let $\alpha, \beta \in [0, 1]$ be the threshold values, against which the values of μ_{I_k,I_l} and ν_{I_k,I_l} are compared. The Criteria I_k and I_l are in:

- (α, β)-positive consonance, if $\mu_{I_k,I_l} > \alpha$ and $\nu_{I_k,I_l} < \beta$;
- (α, β)-negative consonance, if $\mu_{I_k,I_l} < \beta$ and $\nu_{I_k,I_l} > \alpha$;
- (α, β)-dissonance, otherwise.

Certainly, the larger α and/or the smaller β, the less number of criteria may be simultaneously connected with the relation of (α, β)-positive consonance. The most information is carried when either the positive or the negative consonance is as large as possible, while the cases of dissonance are less informative and are skipped in practical purposes. The Table 3.1 gives the threshold values for different types of correlations between the criteria [14].

S.No	Type of Correlations	Degree of Correlations
1	strong positive consonance	$[0.95; 1]$
2	positive consonance	$[0.85; 0.95)$
3	weak positive consonance	$[0.75; 0.85)$
4	weak dissonance	$[0.67; 0.75)$
5	dissonance	$[0.57; 0.67)$
6	strong dissonance	$[0.43; 0.57)$
7	dissonance	$[0.33; 0.43)$
8	weak dissonance	$[0.25; 0.33)$
9	weak negative consonance	$[0.15; 0.25)$
10	negative consonance	$[0.5; 0.15)$
11	strong negative consonance	$[0; 0.5)$

Table 3.1 Scaling for types of correlation between the criteria [14]

10.3.3 ICDM Analysis to the Indian Universities ranking system

The Indian University Ranking System 2017 is presented by the Ministry of Human Resource Development (MHRD) by Government of India under the National Institutional Ranking Framework (NIRF) [122]. Indian Ranking 2017 builds on the previous year's experience, consolidating the Framework, but stand equally challenging and an equally great experience. In [124], Report of Indian Ranking System 2017 contains information on top 17 accredited universities , which offer education in a various disciplines. The ranking system contains informations and data expressed by 17 indicators, which measures different aspects of university activities [125]. The parameters have been choosen in such a manner that these are equally relevant for various kinds of educational institutions [126]. The NIRF provides for ranking of institutes in five broad generic parameters, namely 1) Teaching, Learning & Resources, 2) Research and Professional Practice, 3) Graduation Outcomes, 4) Outreach and Inclusivity, 5) Perception [122]. Each Parameter has its own subparameters which acts as group of indicators. The final assessment is provided in the range from 0 to 100.

The parameters which are used for analysis are listed below [125]:

1. Student Strength including Doctoral Students (SS)
2. Faculty-Student Ratio with emphasis on permanent faculty (FSR)
3. Combined metric for Faculty with PhD (or equivalent) and Experience (FQE)
4. Financial Resources and their Utilisation (FRU)
5. Combined metric for Publications (PU)
6. Combined metric for Quality of Publications (QP)
7. IPR and Patents: Published and Granted (IPR)
8. Footprint of Projects, Professional Practice and Executive Development Programs (FPPP)
9. Metric for University Examinations (GUE)
10. Metric for Number of Ph.D. Students Graduated (GPHD)
11. Percent Students from other states/countries (Region Diversity RD)
12. Percentage of Women (Women Diversity WD)
13. Economically and Socially Challenged Students (ESCS)
14. Facilities for Physically Challenged Students (PCS)
15. Peer Perception: Employers and Research Investors (PREMP)
16. Peer Perception: Academic Peers (PRACD)
17. Public Perception (PRPUB)

In Table 3.2 the number of pairs of criteria for year 2017 for the rating of universities obtained by applying the ICDM method are shown below

Table 3.2 Number of pairs of criteria

Types of correlations	Number of pairs of criteria for the year 2017
positive consonance $[0.85; 0.95)$	1
weak positive consonance $[0.75; 0.85)$	10
weak dissonance $[0.67; 0.75)$	11
dissonance $[0.57; 0.67)$	25
strong dissonance $[0.43; 0.57)$	33
dissonance $[0.33; 0.43)$	23
weak dissonance $[0.25; 0.33)$	24
weak negative consonance$[0.15; 0.25)$	9

The correlation between the indicators in 2017 for the ranking of universities are shown below.

Pair of criteria in positive consonance [0.85;0.95)

- for the year 2017: 2-3;

Pair of criteria in weak positive consonance [0.75;0.85)

- for the year 2017: 2-4, 3-4, 3-11, 4-11, 5-6, 6-15, 6-16, 8-17, 15-16, 15-17;

Pair of criteria in weak dissonance [0.67;0.75)

- for the year 2017: 1-8, 2-11, 5-7, 5-15, 6-8, 6-10, 7-8, 7-15, 7-17, 8-15, 16-17;

Pair of criteria in dissonance [0.57;0.67)

- for the year 2017: 1-6, 1-10, 1-15, 1-17, 2-5, 2-7, 3-5, 3-7, 3-16, 4-5, 4-7, 4-16, 4-17, 5-8, 5-11, 5-16, 5-17, 6-7, 6-17, 7-16, 8-16, 9-13, 10-16, 11-15, 11-16;

Pair of criteria in strong dissonance [0.43;0.57)

- for the year 2017: 1-5, 1-7, 1-12, 1-14, 1-16, 2-8, 2-12, 2-15, 2-16, 2-17, 3-6, 3-8, 3-12, 3-15, 3-17, 4-6, 4-8, 4-12, 4-15, 5-10, 6-11, 7-11, 7-12, 8-10, 8-11, 8-14, 9-12, 10-12, 10-14, 10-15, 10-17, 11-17, 12-14;

Pair of criteria in dissonance [0.33;0.43)

• for the year 2017: 1-13, 2-6, 2-9, 2-13, 2-14, 3-9, 3-10, 3-13, 4-9, 4-10, 6-14, 7-10, 8-12, 8-13, 9-11, 9-17, 10-11, 11-12, 12-13, 12-17, 13-14, 13-15, 13-17;

Pair of criteria in weak dissonance [0.25;0.33)

• for the year: 1-4, 1-11, 2-10, 3-14, 4-13, 4-14, 5-9, 5-12, 6-12, 7-13, 7-14, 8-9, 9-10, 9-14, 9-15, 10-13, 11-13, 11-14, 12-15, 12-16, 13-16, 14-15, 14-16, 14-17;

Pair of criteria in weak negative dissonance [0.15;0.25)

• for the year 2017: 1-2, 1-3, 1-9, 5-13, 5-14, 6-9, 6-13, 7-9, 9-16;

From the comparison of the results over the period of research 2017 the following outcomes are obtained and the correlation shows that whether and how pairs of criteria are related, are given as follows:

• The correlation between them are "positive consonance", "weak negative consonance", "weak dissonance", "dissonance", "strong dissonance", "dissonance", "weak dissonance", "weak negative dissonance".

• There is *no pair of criteria in strong positive consonance.* That is, no strong dependencies which shows that, indicators are well choosen.

• The pair of criteria (2-3) in positive consonance are dependent that is related to each other and are positively correlated.

• There are 10 pairs of criteria in weak positive consonance are dependent that are weakly related to each other.

• There are 33 pairs of criteria in strong dissonance are independent and they are not related to each other.

• There are 9 pairs of criteria in weak negative consonance are negatively correlated that is inverse of weak positive correlation.

Conclusion

In this report, an attempt has been made to introduce intuitionistic fuzzy logic tools. In addition, on the foundation of the theory of intuitionistic fuzzy sets, traditional research is also extended by presenting new definitions and properties of intuitionistic fuzzy statistical tools.

A common architecture of IFLC is designed and the validity of the proposed architecture is clearly verified through the experimental results. In addition, important components of intuitionistic fuzzy logic controller namely intuitionistic fuzzification and intuitionistic defuzzification functions are also defined with suitable illustrations.

Intuitionistic fuzzy random variable is defined and some of its properties are discussed. IF statistical tools like mean, median, mode for IF data defined are very helpful for developing IF filters in image processing. Four different filtering techniques namely IF mean, IF median, IF maximum, IF minimum filters are defined and their filtering performance on impulse noise is presented. The performance of the proposed IF filtering technique is evaluated in MATLAB simulations for an image that has been subjected to various degrees of corruption with impulse noise. The results demonstrate the effectiveness of the algorithm.

A new approach has been introduced using intuitionistic fuzzy moving average and compared with existing crisp and fuzzy moving average operators. Also effectiveness of the proposed moving average technique is verified with an numerical data set to make decisions on GDP growth in India.

The concept of distance, center, eccentricity of an intuitionistic fuzzy tree is introduced. The procedure for intuitionistic fuzzification for numerical data set is proposed. This report also provides intuitionistic fuzzy tree center based clustering techniques for numerical data set with multiple attributes to produce clusters. The algorithm is tested on a data set containing information of 27 nutrients with five features and implemented on MATLAB.

Some interesting properties of IFDHGs are dealt with p-coloring, $\mathcal{K}$-coloring, p-chromatic number, spike, spike reduction and skeleton of spike reduction. Further, it has been proved that if H is an ordered IFDHG and A is a primitive coloring of H, then A is a $\mathcal{K}$-coloring of H and some other properties have also been analysed.

Chromatic values and chromatic numbers of intuitionistic fuzzy colorings, upper and lower truncation, core aggregate, conservative $\mathcal{K}-$ coloring of intuitionistic fuzzy directed hypergraph, elementary center of intuitionistic fuzzy coloring, f-chromatic value of intuitionistic fuzzy coloring intersecting IFDHG, $\mathcal{K}-$ intersecting IFDHG, strongly intersecting IFDHG were studied. Also it has been proved that IFDHG H is strongly intersecting if and only if it is $\mathcal{K}-$ intersecting.

Essentially intersecting, essentially strongly intersecting, skeleton intersecting, non-trivial, sequentially simple and essentially sequentially simple IFDHG are defined. Also an application of IFDHGs in molecular structure representation has been given. As this is an initiative taken to represent molecular structures using IFDHGs, the authors further proposed to apply the properties of IFDHGs to study and compare the properties of molecular structures of all states of water.

Multi-parameter temporal intuitionistic fuzzy set is defined as a special case of IFMDS defined in [17] which takes into account the possibility of different parameter sets in a TIFS. MTIFS is of great significance as it is a useful tool in systems with different time domains and with multiple parameters. Some operators on MTIFSs are also defined. As intuitionistic fuzzification function is the first step to design intuitionistic fuzzy logic control systems, extended triangular intuitionistic fuzzification functions are defined for TIFSs and MTIFSs. Further, the authors proposed to work on other types of intuitionistic fuzzification functions of TIFSs and MTIFSs and their applications in dynamic systems.

Finally, the Inter Criteria Decision Making (ICDM) method is used to find some hidden patterns in the data using Indian Ranking 2017 [122]. Indian Ranking for the year 2017 contains 17 Universities and 17 indicators which are used to analyze the data to identify the best correlation between the indicators and the relationship between them.

It is inferred that no pairs of indicators are in strong positive consonance and thus it leads to non-removal of the criteria in ranking system. In future, the researchers may consider all pairs of criteria and if there exist any pairs of criteria in strong positive consonance which will lead to removal of one of the criteria in the data which has the less informational values. The simplification of the process of evaluation is due to removal of indicators.

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OUTPUT OF THE PROJECT

Details of Publications

S. No.	Title of the paper	Author(s)	Journal name	Year, Vol.(Issue), Page No.
1	Intuitionistic fuzzy random variable	R. Parvathi and C. Radhika	Notes on Intuitionistic Fuzzy Sets (indexed in Directory of Research Journals Indexing (since 2016), EZB (Electronic Journals Library) (since 2016), Index Copernicus (in 2012, for 2015 under evaluation), ROAD – Directory of Open Access Scholarly Resources (since 2015), WorldCat / OCLC (since 2016) and Zentralblatt MATH (since 1995)	2015, 21(1), 69 – 80
2	Intuitionistic fuzzification functions	C. Radhika and R. Parvathi	Global Journal of Pure and Applied Mathematics (indexed in the Mathematical Reviews, MathSciNet, Zentralblatt MATH and EBSCO Databases, ICI, Index Copernicus)	2016, 12(2), 1211 – 1227
3	Defuzzification of intuitionistic fuzzy sets	C. Radhika and R. Parvathi	Notes on Intuitionistic Fuzzy Sets (indexed in Directory of Research Journals Indexing (since 2016), EZB (Electronic Journals Library) (since 2016), Index Copernicus (in 2012, for 2015 under evaluation), ROAD – Directory of Open Access Scholarly Resources (since 2015), WorldCat / OCLC (since 2016) and Zentralblatt MATH (since 1995)	2016, 22(5), 19 – 26
4	Noise removal from images using intuitionistic fuzzy logic controller	C. Radhika, R. Parvathi and N. Karthikeyani Visalakshi	Annals of Fuzzy Mathematics and Informatics (indexed in MatheScinet and Korea Citation Index, and UGC approved journal)	2017,13(3), 421 – 435
5	Intuitionistic Fuzzy Tree Center-Based Clustering Algorithm	G.Thamizhendhi and R. Parvathi	International Journal of Soft Computing and Engineering B Impact Factor: 5.23 (Year 2017)	2016, 6(1), 50 – 65

6	Chromatic Values of Intuitionistic Fuzzy Directed Hypergraph Colorings	K. K. Myithili and R. Parvathi	International Journal of Soft Computing and Engineering B Impact Factor: 5.23 (Year 2017)	2016, 6(1), 32 – 37
7	Coloring of Intuitionistic Fuzzy Directed Hypergraphs	K. K. Myithili and R. Parvathi	International Journal of Computer Application Universal Impact Factor=0.45	2016, 6(3), 159 – 166
8	Intuitionistic Fuzzy Filters for Noise Removal in Images	C. Radhika, R. Parvathi and N. Karthikeyani Visalakshi	Handbook of Research on Generalized and Hybrid Set Structures and Applications for Soft Computing	2016, 53, 54 – 64
9	Multi-parameter temporal intuitionistic fuzzy sets	R. Parvathi and C. Radhamani	Notes on Intuitionistic Fuzzy Sets (indexed in Directory of Research Journals Indexing (since 2016), EZB (Electronic Journals Library) (since 2016), Index Copernicus (in 2012, for 2015 under evaluation), ROAD – Directory of Open Access Scholarly Resources (since 2015), WorldCat / OCLC (since 2016) and Zentralblatt MATH (since 1995)	2016, 22(1), 35 – 47
10	An application of intuitionistic fuzzy directed hypergraph in molecular structure representation	R.Parvathi, C.Yuvapriya and N.Maragatham	Notes on Intuitionistic Fuzzy Sets (indexed in Directory of Research Journals Indexing (since 2016), EZB (Electronic Journals Library) (since 2016), Index Copernicus (in 2012, for 2015 under evaluation), ROAD – Directory of Open Access Scholarly Resources (since 2015), WorldCat / OCLC (since 2016) and Zentralblatt MATH (since 1995)	2017, 23(1), 69 – 78
11	InterCriteria Analysis of rankings of Indian universities	R. Parvathi, Vassia Atanassova, Lyubka Doukovska, C. Yuvapriya and K. Indhurekha	Notes on Intuitionistic Fuzzy Sets (indexed in Directory of Research Journals Indexing (since 2016), EZB (Electronic Journals Library) (since 2016), Index Copernicus (in 2012, for 2015 under evaluation), ROAD – Directory of Open Access Scholarly Resources (since 2015), WorldCat / OCLC (since 2016) and Zentralblatt MATH (since 1995)	2018, 24(1), 99 – 109

Details of Paper Presentations

S. No.	Date	Title of the Presentation/Poster	Conference/Seminar/ Workshop	Host Institute
Dr.R.Parvathi, Principal Investigator				
1	22.05.2017 & 23.05.2017	An application of IFDHGs in molecular structure representation	21st International Conference on Intuitionistic Fuzzy Sets (ICIFS'2017)	Burgas University "Prof.Assen Zlatarov", Burgas, Bulgaria
2	08.01.2018 to 10.01.2018	InterCriteria Analysis of rankings of Indian universities	International Conference on Intuitionistic fuzzy Sets and Systems (ICIFSS – 2018)	Vellalar College for Women, Erode, Tamilnadu.
Ms.C.Yuvapriya, Project Fellow				
3	17.03.2017 & 18.03.2017	Intersecting Intuitionistic Fuzzy Directed Hypergraphs	International Conference on Applied Mathematics and Informatics (ICAMI 2017)	Kongu Engineering College, Perundurai, Erode, Tamilnadu.
4	02.01.2018 & 03.01.2018	Intuitionistic Fuzzy Directed hypergraph theory in Chemistry	Indian Women and Mathematics - Regional workshop on Research and Opportunities	Cochin University of Science and Technology, Kerala.
5	08.01.2018 to 10.01.2018	Dual Intuitionistic fuzzy directed hypergraph	International Conference on Intuitionistic fuzzy Sets and Systems (ICIFSS – 2018)	Vellalar College for Women, Erode, Tamilnadu.

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Intuitionistic Fuzzy (IF) Logic Toolbox

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Intuitionistic Fuzzy (IF) Logic Toolbox helps to plot the standard membership and non-membership functions.

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iftrif
Not enough input arguments.
Error in iftrif (line 3)

Answer: After assigning value to x, the following syntax must be followed with x as the first parameter.
[y,z]=iftrif(x,a,b,c,e)

said broumi
24 Jul 2018

iftrif
Not enough input arguments.

Error in iftrif (line 3)
y=zeros(1,length(x));

COULD YOU PLEASE HOW TO RESOLVE THIS PROBLEM

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```
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function [y,z]=iftrif(x,a,b,c,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=0;
    z(j)=1-e;
elseif(x(j)>a)&&(x(j)<=b)
```

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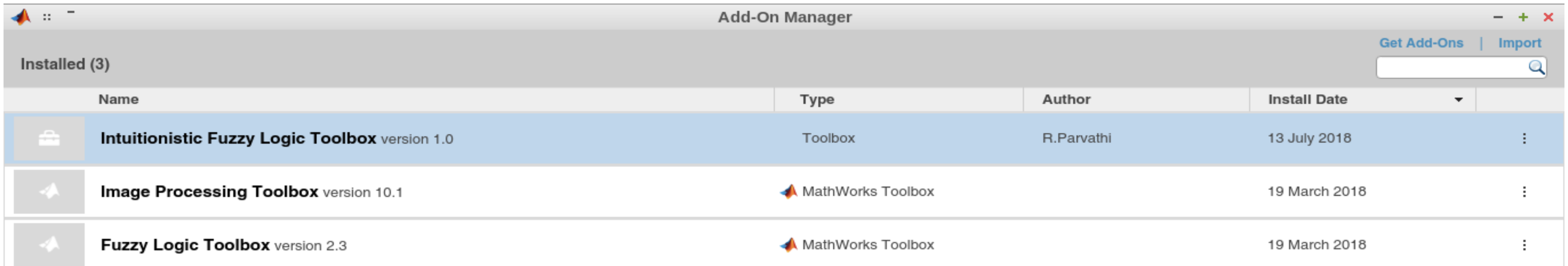
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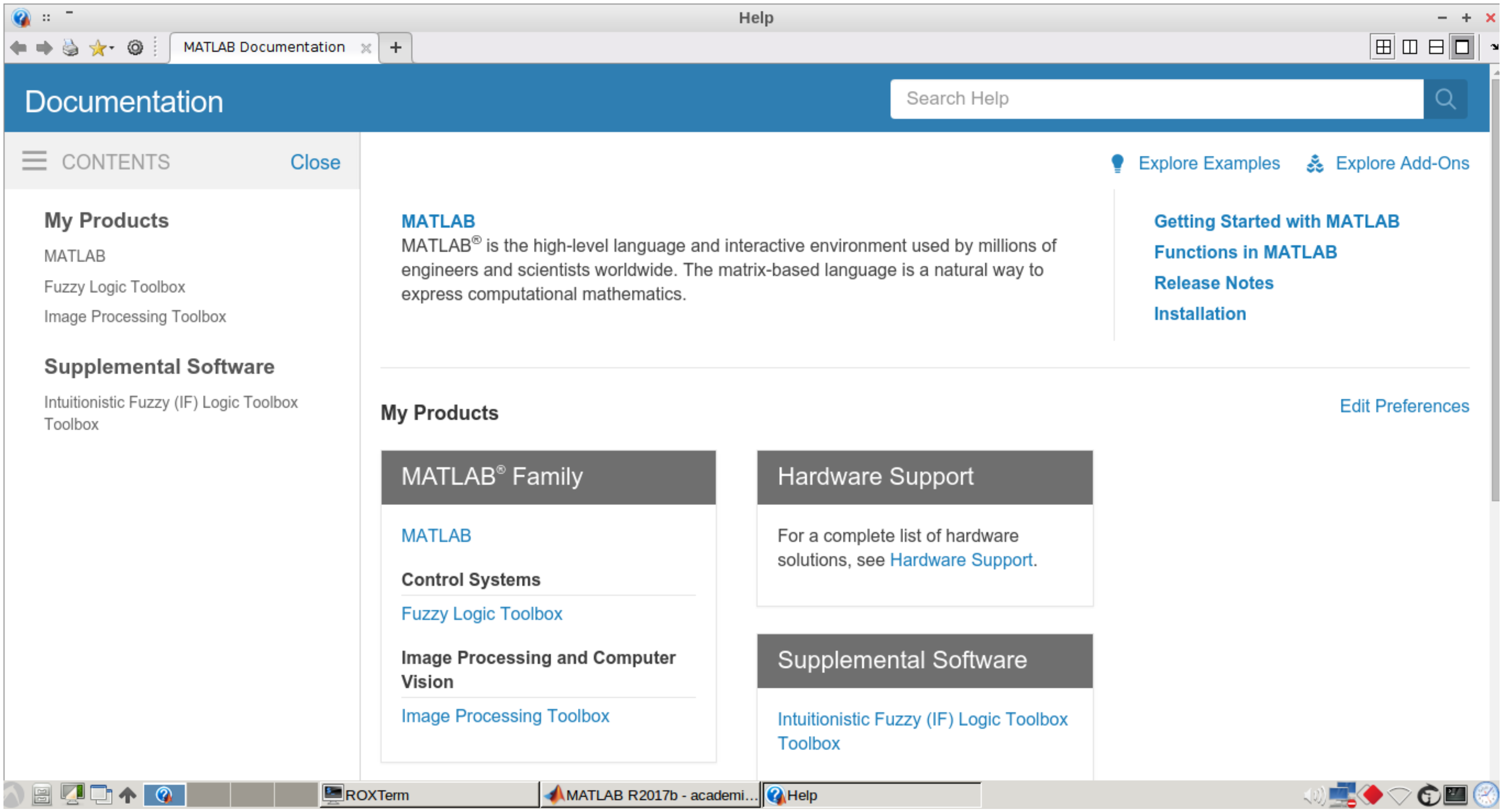
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Intuitionistic Fuzzy Logic Toolbox: Coding and output

Intuitionistic Fuzzy Triangular Function (iftrif)

M - File Coding:

```
function [y,z]=iftrif(x,a,b,c,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=0;
    z(j)=1-e;
elseif(x(j)>a)&&(x(j)<=b)
    y(j)=((x(j)-a)/(b-a))-e;
    z(j)=1-((x(j)-a)/(b-a));
elseif(x(j)>=b)&&(x(j)<c)
     y(j)=((c-x(j))/(c-b))-e;
     z(j)=1-((c-x(j))/(c-b));
elseif(x(j)>=c)
    y(j)=0;
    z(j)=1-e;
end
end
```

Command Window coding:

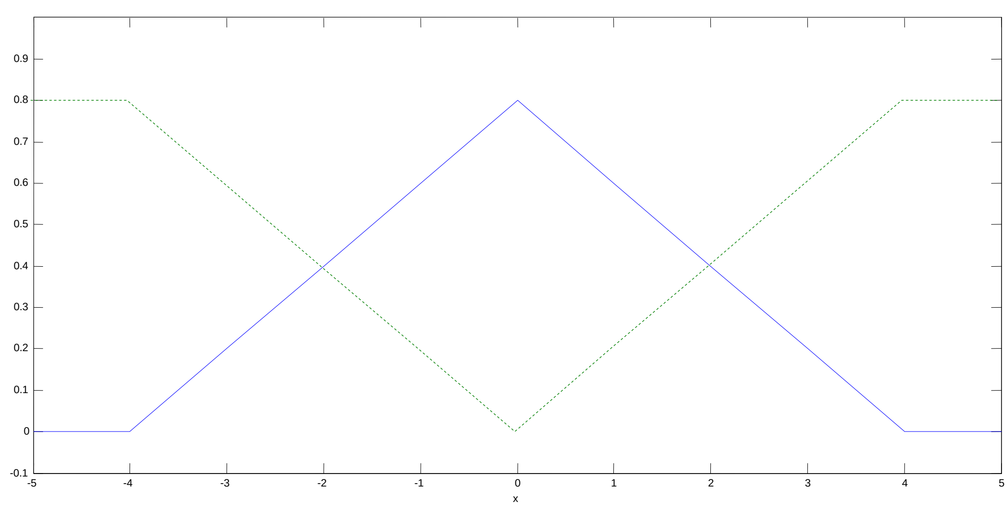
```
>> x=-5:5;
>> x
x =

    -5     -4     -3     -2     -1      0      1      2      3      4      5
>> [y,z]=iftrif(x,-5,0,5,0.2);
>> plot(x,y,x,z)
>>
```

Output:

——— Membership function

- - - - -Non Membership function



Intutionistic Fuzzy Trapezoidal Function (iftraf)

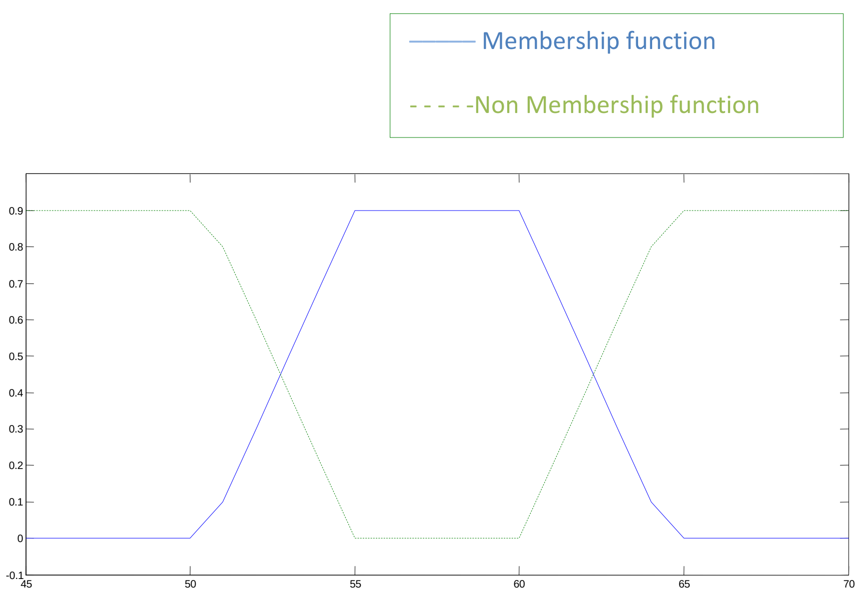
M - File Coding:

```
function [y,z]=iftraf(x,a,b,c,d,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=0;
    z(j)=1-e;
elseif(x(j)>a)&&(x(j)<b)
    y(j)=((x(j)-a)/(b-a))-e;
    z(j)=1-((x(j)-a)/(b-a));
elseif(x(j)>=b)&&(x(j)<=c)
    y(j)=1-e;
    z(j)=0;
elseif(x(j)>c)&&(x(j)<d)
     y(j)=((d-x(j))/(d-c))-e;
     z(j)=1-((d-x(j))/(d-c));
elseif(x(j)>=d)
    y(j)=0;
    z(j)=1-e;
end
end
end
```

Command Window coding:

```
>> x=45:70;
>> [y,z]=iftraf(x,50,55,60,65,0.1);
>> plot(x,y,x,z)
```

Output:



Intutionistic Fuzzy R-Function (ifRf)

M - File Coding:

```
function [y,z]=ifRf(x,c,d,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)>=d)
    y(j)=0;
    z(j)=1-e;
elseif(x(j)>c)&&(x(j)<d)
     y(j)=((d-x(j))/(d-c))-e;
     z(j)=1-((d-x(j))/(d-c));
elseif(x(j)<=c)
    y(j)=1-e;;
    z(j)=0;
end
end
end
```

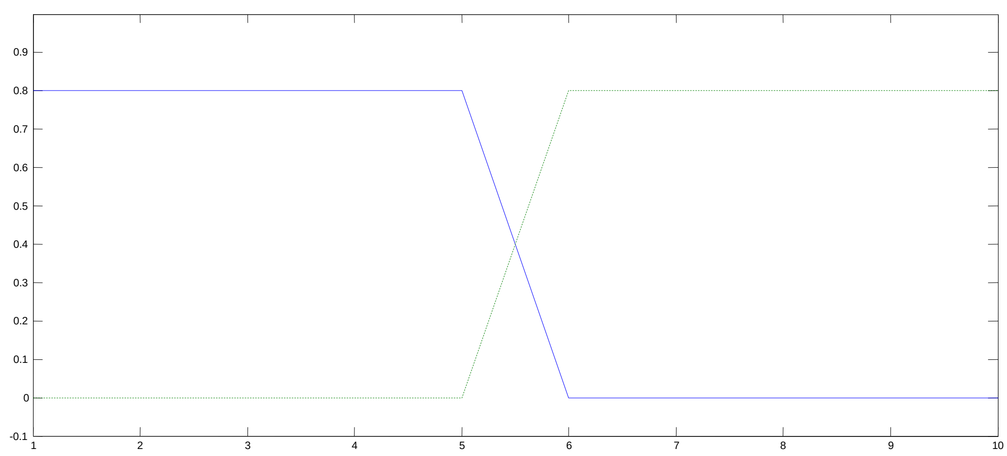
Command Window coding:

```
>> x=1:10;
>> [y,z]=ifRf(x,5.6,5.8,0.2);
>> plot(x,y,x,z)
```

——— Membership function

- - - - -Non Membership function

Output:



Intutionistic Fuzzy L-Function (ifLf)

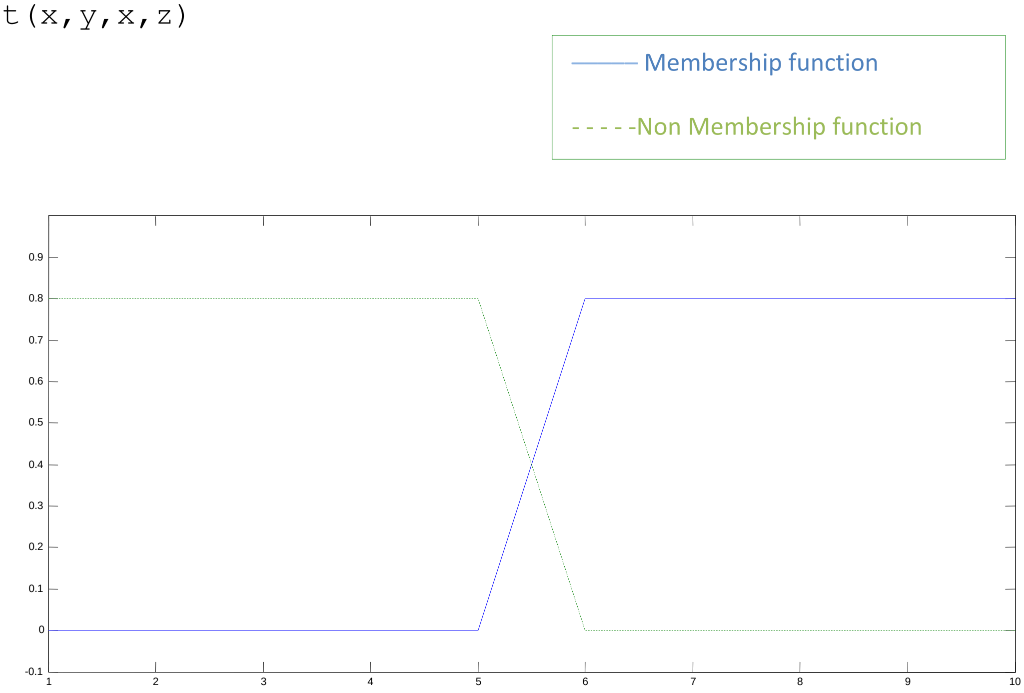
M - File Coding:

```
function [y,z]=ifLf(x,a,b,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=0;
    z(j)=1-e;
elseif(x(j)>a)&&(x(j)<b)
    y(j)=((x(j)-a)/(b-a))-e;
    z(j)=1-((x(j)-a)/(b-a));
elseif(x(j)>=b)
    y(j)=1-e;
    z(j)=0;
end
end
end
```

Command Window coding:

```
>> x=1:10;
>> [y,z]=ifLf(x,5.2,5.4,0.2);
>> plot(x,y,x,z)
```

Output:



Intuitionistic Fuzzy Gaussian Function (ifgaussf)

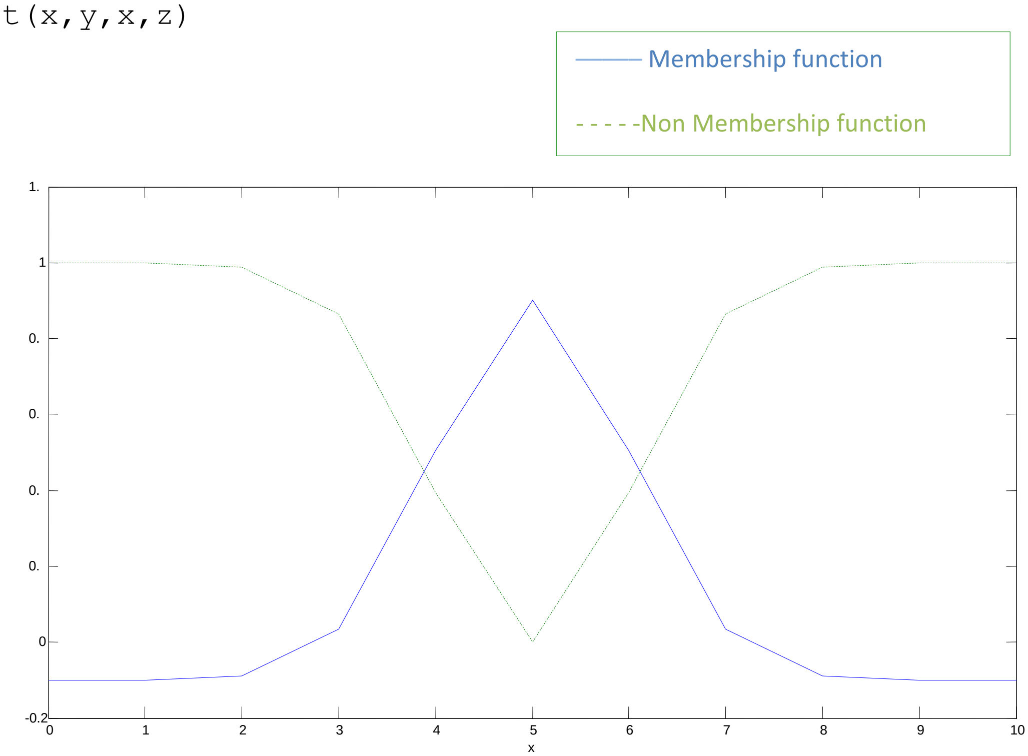
M - File Coding:

```
function [y,z]=ifgaussf(x,m,k,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
y(j)=exp((-(x(j)-m)^2)/2*k^2)-e;
z(j)=1-exp((-(x(j)-m)^2)/2*k^2);
end
end
```

Command Window coding:

```
>> x=0:10;
>> [y,z]=ifgaussf(x,5,1,0.1);
>> plot(x,y,x,z)
```

Output:



Intuitionistic Fuzzy Bell-shaped Function (ifbellf)

M - File Coding:

```
function [y,z]=ifbellf(x,a,b,c,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
    y(j)=1-e-(1/(1+(abs((x(j)-c)/a))^(2*b)));
    z(j)=1/(1+(abs((x(j)-c)/a))^(2*b));
end
```

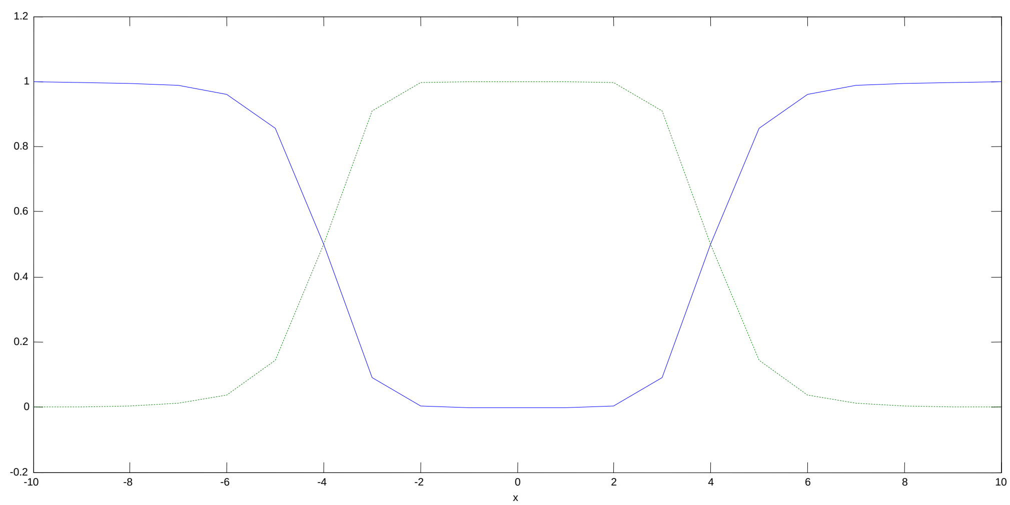
Command Window coding:

```
>>x=-10:10;
>> [y,z]=ifbellf(x,-4,4,0,0.001);
>> plot(x,y,x,z)
```

——— Membership function

- - - - -Non Membership function

Output:



Intuitionistic Fuzzy S-shaped Function (ifSf)

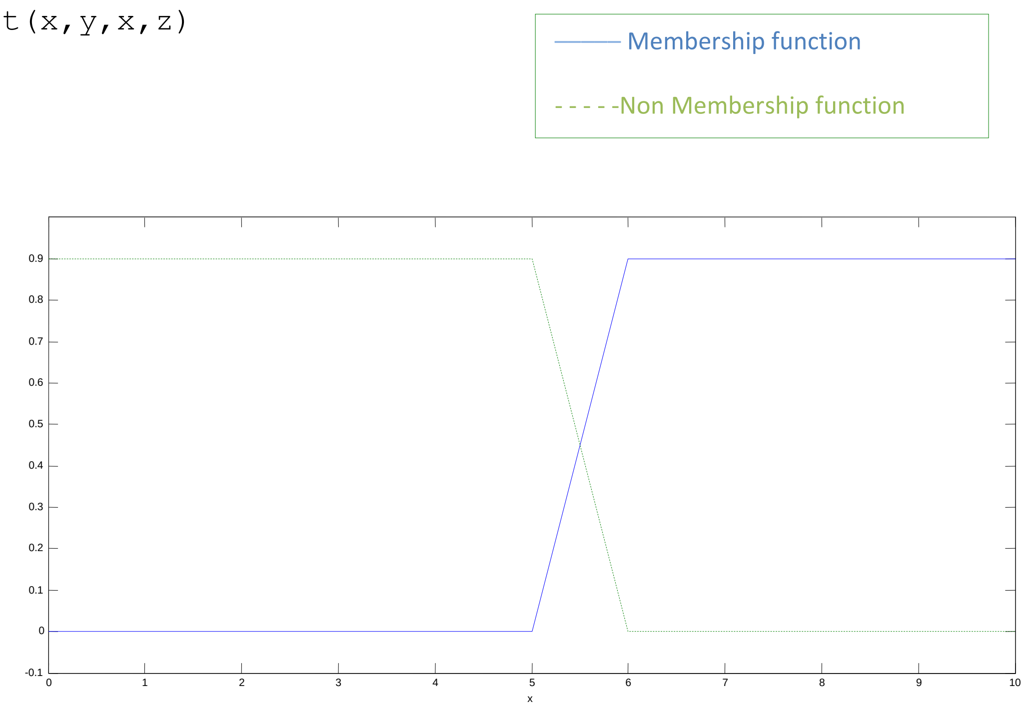
M - File Coding:

```
function [y,z]=ifSf(x,a,b,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=0;
    z(j)=1-e;
elseif(x(j)>a)&&(x(j)<=((a+b)/2))
    y(j)=2*(((x(j)-a)/(b-a))^2)-e;
    z(j)=1-(2*(((x(j)-a)/(b-a))^2));
elseif(x(j)>=((a+b)/2))&&(x(j)<b)
    y(j)=1-(2*(((x(j)-a)/(b-a))^2))-e;
    z(j)=2*(((x(j)-a)/(b-a))^2);
elseif(x(j)>=b)
     y(j)=1-e;
     z(j)=0;
end
end
```

Command Window coding:

```
>> x=0:10;
>> [y,z]=ifSf(x,5.1,5.5,0.1);
>> plot(x,y,x,z)
```

Output:



Intuitionistic Fuzzy Z-shaped Function (ifZf)

M - File Coding:

```
function [y,z]=ifZf(x,a,b,e)
y=zeros(1,length(x));
z=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=1-e;
    z(j)=0;
elseif(x(j)>a)&&(x(j)<=((a+b)/2))
    y(j)=1-(2*(((x(j)-a)/(b-a))^2))-e;
    z(j)=2*(((x(j)-a)/(b-a))^2);

elseif(x(j)>=((a+b)/2))&&(x(j)<b)
     y(j)=2*(((x(j)-a)/(b-a))^2)-e;
     z(j)=1-(2*(((x(j)-a)/(b-a))^2));
elseif(x(j)>=b)
     y(j)=0;
     z(j)=1-e;
end
end
```

Command Window coding:

```
>> x=0:10;
>> [y,z]=ifZf(x,5.1,5.5,0.1);
>> plot(x,y,x,z)
```

——— Membership function

- - - - -Non Membership function

Output: